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Definite Quadratic Eigenvalue Problems

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Abstract

Free vibrations of fluid-solids structures are governed by a nonsymmetric eigenvalue problem. This problem can be transformed into a definite quadratic eigenvalue problem. The present paper considers properties of definite problems and variational characterization for definite quadratic eigenvalue problem. We propose improvement of existing methods which determine whether a quadratic pencil $\mathbf{Q}(\lambda)$ is definite. The improved method is reflected in a better localization of start parameters μ and ξ where $\mathbf{Q}(\mu) > 0 > \mathbf{Q}(\xi)$ and $\mu > \xi$.

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Key words: free vibrations; definite quadratic pencil; eigenvalue; variational characterization; methods for detection

1. Introduction

In practice, nonlinear eigenproblems commonly arise in dynamic/stability analysis of structures and in fluid mechanics, electronic behavior of semiconductor hetero-structures, vibration of fluid-solid structures, vibration of sandwich plates, accelerator design, vibro-acoustics of piezoelectric/poroelestic structures, nonlinear integrated optics, regularization on total least squares problems and stability of delay differential equations. In practice, the most important is the quadratic problem $\mathbf{Q}(\lambda)\mathbf{x} = \mathbf{0}$ where

$$\mathbf{Q}(\lambda) \coloneqq \lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}, \qquad \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{C}^{n\times n}, \qquad \mathbf{A} \neq \mathbf{0}, \tag{1}$$

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([11] for a recent survey). The theoretical basics for the quadratic eigenvalue problems are given in the [7].

For quadratic hyperbolic pencils, Higham, Tisseur and Van Dooren [5] proposed a method for testing hyperbolicity and for constructing a definite linearization. Another method for detecting whether a Hermitian quadratic matrix polynomial is hyperbolic is based on cyclic reduction and was first introduced by Guo and Lancaster [3] and later on, accelerated by Guo, Higham and Tisseur [2].

Niendorf and Voss [9] take advantage of the fact that all eigenvalues of a definite matrix polynomial can be characterized as minmax values of appropriate Rayleigh functional and that the extreme eigenvalues in each of the intervals $(-\infty, \xi)$, (ξ,μ) and $(\mu,+\infty)$ are the limits of monotonically and quadratically convergent sequences. If a given Hermitian quadratic matrix polynomial is definite, Niendorf and Voss concurrently determine parameters ξ and μ such that the matrices $\mathbf{Q}(\xi) < 0$ and $\mathbf{Q}(\mu) > 0$, which allows for the computation of a definite linearization.

Kostić and Voss [8] applied the Sylvester's law of inertia on definite quadratic eigenvalue problems.

In this paper we improve the method of Niendorf and Voss [9] for detecting whether a quadratic pencil $\mathbf{Q}(\lambda)$ is definite. The improvement is reflected in a better localization of the parameters μ and ξ where $\mathbf{Q}(\mu) > 0 > \mathbf{Q}(\xi)$, and $\mu > \xi$. The method in [9] needs a lot of flops in every step of the given algorithm, consequently this takes a lot of computational time and is subject to numerical instabilities. This is motivation for improvement of already existing algorithm. Our improvement is in a systematic determination of the initial vector $\mathbf{x}_0$, thus reducing the number of steps in the algorithm. In this paper we also bring the theoretical basics which enable such selection of the initial vector $\mathbf{x}_0$.

Our paper is organized as follows. In Section 2 we provide the basic terms and their properties: hyperbolic quadratic pencil and definite quadratic pencil. Section 3 gives a practical example of how the free vibrations of fluid-solids structures can be described with the definite quadratic polynomial. In Section 4 we propose an improvement of the Niendorf and Voss method and show the theoretical basics for an improved initial location for ξ and μ parameters. A critical review of the proposed method is given in Section 5.

2. Quadratic problems

In this paragraph we give the theoretical basics for the hyperbolic quadratic pencil and the definite quadratic pencil.

2.1. Hyperbolic quadratic pencil

A quadratic matrix polynomial

$$\mathbf{Q}(\lambda) := \lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}, \qquad \mathbf{A} = \mathbf{A}^H > 0, \qquad \mathbf{B} = \mathbf{B}^H, \;\; \mathbf{C} = \mathbf{C}^H \tag{2}$$

is hyperbolic if for every $\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq \mathbf{0}$ the quadratic polynomial

$$\mathbf{f}(\lambda; \mathbf{x}) := \lambda^2\mathbf{x}^H\mathbf{A}\mathbf{x} + \lambda\mathbf{x}^H\mathbf{B}\mathbf{x} + \mathbf{x}^H\mathbf{C}\mathbf{x} = \mathbf{0} \tag{3}$$

has two distinct real roots:

$$p_+(\mathbf{x}) := -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}} + \sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{A}\mathbf{x}}}\,, \qquad p_-(\mathbf{x}) := -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}} - \sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{A}\mathbf{x}}}\,. \tag{4}$$

The functional in Eq. 4 are the Rayleigh functionals of the quadratic matrix polynomial (Eq. 2). The Rayleigh functionals are the generalization of the Rayleigh quotient.

The ranges $J_+ := p_+(\mathbb{C}^n\backslash\{\mathbf{0}\})$ and $J_- := p_-(\mathbb{C}^n\backslash\{\mathbf{0}\})$ are disjoint real intervals with $\max J_- < \min J_+$.

$\mathbf{Q}(\lambda)$ is the positive definite for $\lambda < \min J_-$ and $\lambda > \max J_+$, and it is the negative definite for $\lambda \in (\max J_-, \min J_+)$.

(Q, J_+) and $(-Q, J_-)$ satisfy the conditions of the variational characterization of the eigenvalues, i.e. there exist $2n$ eigenvalues [1].

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n < \lambda_{n+1} \le \cdots \le \lambda_{2n} \tag{5}$$

and

$$\lambda_j = \min_{\dim V=j} \max_{x\in V, x\neq 0} p_-(\mathbf{x}), \qquad \lambda_{n+j} = \min_{\dim V=j} \max_{x\in V, x\neq 0} p_+(\mathbf{x}), \qquad j = 1,2,\cdots,n. \tag{6}$$

2.2. Definite quadratic pencils

Higham, Mackey and Tisseur [6] generalized the concept of the hyperbolic quadratic polynomials by waiving the positive definiteness of te leading matrix **A.**
A quadratic pencil (Eq. 1) is definite if:
$\mathbf{A} = \mathbf{A}^H, \mathbf{B} = \mathbf{B}^H, \ \mathbf{C} = \mathbf{C}^H$ are Hermitian,
there exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that $\mathbf{Q}(\mu)$ is positive definite, and
for every fixed $\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq \mathbf{0}$ the quadratic polynomial (3) has two distinct roots in $\mathbb{R} \cup \{\infty\}$.
The following Theorem has been proved by Duffin [1].
Theorem 1. A Hermitian matrix polynomial $\mathbf{Q}(\lambda)$ is definite if and only if any two (and hence all) of the following properties hold:

- $d(\mathbf{x}) := (\mathbf{x}^H\mathbf{B}\mathbf{x})^2 - 4(\mathbf{x}^H\mathbf{A}\mathbf{x})(\mathbf{x}^H\mathbf{C}\mathbf{x}) > \mathbf{0}$ for every $\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq \mathbf{0}$
- $\mathbf{Q}(\mu) > 0$ for same $\mu \in \mathbb{R} \cup \{\infty\}$
- $\mathbf{Q}(\xi) < 0$ for same $\xi \in \mathbb{R} \cup \{\infty\}$.

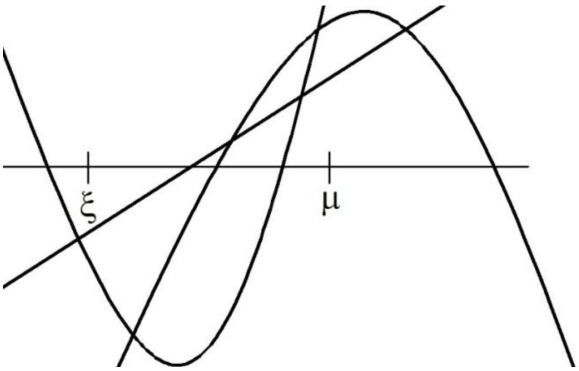

Fig. 1. Definite quadratic matrix polynomial $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$, $\xi < \mu$..

Without loss of generality, we take that $\xi < \mu$. Fig. 1 illustrates equations:

$f(\mu; \mathbf{x}) = \mathbf{x}^H\mathbf{Q}(\mu)\mathbf{x}$
$f(\xi; \mathbf{x}) = \mathbf{x}^H\mathbf{Q}(\xi)\mathbf{x}$
where $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$, $\xi < \mu$.
The Rayleigh functional is explicitly given by

$$p(\mathbf{x}) := \begin{cases} -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}} + \sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{A}\mathbf{x}}} & \text{if } \mathbf{x}^H\mathbf{A}\mathbf{x} > 0 \\ -\frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{B}\mathbf{x}} \text{ if } & \mathbf{x}^H\mathbf{A}\mathbf{x} = 0 \\ -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}} - \sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{A}\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{A}\mathbf{x}}} & \text{if } \mathbf{x}^H\mathbf{A}\mathbf{x} < 0 \end{cases}$$

The quadratic eigenvalue problem $\mathbf{Q}(\lambda)\mathbf{x} = \mathbf{0}$ has exactly n eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ in (ξ, μ) which satisfy a minmax characterization with respect to p, and the safeguarded iterations aiming at λ_1 and λ_n converge globally and monotonically decreasing and increasing, respectively. Niendorf and Voss [9] gave the following algorithm (Fig. 2) for detecting definite quadratic matrix polynomial.

Require: initial vector $\mathbf{x}_0 \neq \mathbf{0}$
if $d(\mathbf{x}_0) := (\mathbf{x}_0^H\mathbf{B}\mathbf{x}_0)^2 - 4(\mathbf{x}_0^H\mathbf{A}\mathbf{x}_0)(\boldsymbol{x}_0^H\boldsymbol{C}\boldsymbol{x}_0) < 0$ **then**
STOP: $\mathbf{Q}(\lambda)$ is not definite
end if
determine $\sigma_0 = p(\mathbf{x}_0)$
for $k = 1, 2, \ldots$ **until convergence do**
determine eigenvector x_k of $\mathbf{Q}(\sigma_{k-1})$ corresponding to its largest eigenvalue
if $d(\mathbf{x}_k) := \left(\mathbf{x}_k^H\mathbf{B}\mathbf{x}_k\right)^2 - 4(\mathbf{x}_k^H\mathbf{A}\mathbf{x}_k)(\mathbf{x}_k^H\mathbf{C}\mathbf{x}_k) < 0$ **then**
STOP: $\mathbf{Q}(\lambda)$ is not definite
end if
determine $\sigma_k = p(\mathbf{x}_k)$
if $\sigma_k \geq \sigma_{k-1}$ **then**
STOP: $\mathbf{Q}(\lambda)$ is not definite
end if
if $Q(2\sigma_k - \sigma_{k-1})$ is negative definite then
STOP: $\mathbf{Q}(\lambda)$ is definite
end if
end for

Fig. 2.Algorithm 1.

If $\mathbf{Q}(\lambda)$ is definite we can determine by Algorithm 1 a parameter ξ such that $\mathbf{Q}(\xi)$ is negative definite. We have to allow for one violation of the monotonicity requirement to incorporate the possible jump of the iterations from one unbounded interval to the other. A second similar sweep aiming at λ_n discovers a parameter μ such that $\mathbf{Q}(\mu) > 0$. Once parameters ξ and μ are found such that $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$, the definite linearization can be determined similarly.

The kth step of Algorithm 1 requires $n^3/3$ operations for computing 1 Cholesky factorization, $4n^2$ operations for evaluating $\mathbf{Q}(2\sigma_k - \sigma_{k-1})$, 3 matrix-vector products ($6n^2$ operations), 3 scalar products ($6n$ operations), and the determination of the largest eigenvalue and corresponding eigenvector of a matrix $\mathbf{Q}(\sigma_{k-1})$ which seems to be the most expensive part.

3. Free vibration of a fluid-solid structures

Free vibrations of a fluid-solids structures are governed by a non-symmetric eigenvalue problem [4,8,10]

$$\begin{pmatrix} \mathbf{K}_S & \mathbf{C} \\ \mathbf{0} & \mathbf{K}_f \end{pmatrix}\begin{pmatrix} \mathbf{x}_s \\ \mathbf{x}_f \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{M}_S & \mathbf{0} \\ -\mathbf{C}^T & \mathbf{M}_f \end{pmatrix}\begin{pmatrix} \mathbf{x}_s \\ \mathbf{x}_f \end{pmatrix}, \tag{7}$$

where $\mathbf{K}_S \in \mathbb{R}^{s\times s}$ and $\mathbf{K}_f \in \mathbb{R}^{f\times f}$ are the stiffness matrices, and $\mathbf{M}_S \in \mathbb{R}^{s\times s}$ and $\mathbf{M}_f \in \mathbb{R}^{f\times f}$ are the mass matrices of the structure and the fluid, respectively, and $\mathbf{C} \in \mathbb{R}^{s\times f}$ describes the coupling of structure and fluid. $\mathbf{x}_s$ is the structure displacement vector, and $\mathbf{x}_f$ is the fluid pressure vector. $\mathbf{K}_s, \mathbf{M}_s, \mathbf{K}_f$ and $\mathbf{M}_f$ are symmetric and positive definite.

This problem can be transformed into a definite quadratic eigenvalue problem. Multiplying the first line of (7) by λ one obtains the definite quadratic pencil

$$\mathbf{Q}(\lambda) := \lambda^2 \begin{pmatrix} \mathbf{M}_S & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \lambda \begin{pmatrix} -\mathbf{K}_S & -\mathbf{C} \\ -\mathbf{C}^T & \mathbf{M}_f \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_f \end{pmatrix}. \tag{8}$$

Example above is seen in the tecnical practice and it is motivation for further consideration of the definite quadratic pencil and eigenvalue problems for these pencils.

4. New strategy

In order to accelerate Algorithm 1 we consider a better localization of the parameters μ and ξ which hold $\mathbf{Q}(\mu) > 0 > \mathbf{Q}(\xi)$ and $\mu > \xi$. From this localization we will get initial vector $\mathbf{x}_0$. For better localization of the mentioned parameters, we will use the essential characteristics of the matrix **A**, **B** and **C**, which define the square pencil $\mathbf{Q}(\lambda) := \lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}$.

Let a quadratic matrix polynomial

$$\mathbf{Q}(\lambda) := \lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}, \qquad \mathbf{A} = \mathbf{A}^H, \qquad \mathbf{B} = \mathbf{B}^H, \;\; \mathbf{C} = \mathbf{C}^H \tag{9}$$

is definite.

The matrix $\mathbf{A} \neq 0$ must be singular, otherwise multiplying Eq. (9) to the left side with $\mathbf{A}^{-1}$ we would get the quadratic pencils

$$\mathbf{Q}_1(\lambda) := \lambda^2\mathbf{I} + \lambda\mathbf{B}_1 + \boldsymbol{C}_1, \;\mathbf{B}_1 := \mathbf{A}^{-1}\mathbf{B}, \;\; \mathbf{C}_1 := \mathbf{A}^{-1}\mathbf{C}\,. \tag{10}$$

Lemma 1 Problem of the eigenvalues corresponding to the matrix polynomial (9) has 0 as an eigenvalue if and only if **C** is singular.

Proof
Let **C** be singular matrix. Then there is vector **z** such that

$$\mathbf{Cz} = \mathbf{0}\,. \tag{11}$$

Than we can apply

$$\mathbf{Q}(0) = \mathbf{Cz} = \mathbf{0}. \tag{12}$$

In this case 0 is eigenvalue of the matrix $\mathbf{Q}(\lambda)$ and **z** corresponding eigenvector. Let now 0 be eigenvalue of the quadratic matrix polynomial (9). Then there is vector **z** such that

$$\mathbf{Q}(0)\mathbf{z} = \mathbf{0} \Rightarrow \mathbf{Cz} = \mathbf{0}, \tag{13}$$

which means that **C** is singular. ∎

Based on Lemma 1 $\mathbf{C}$ must be singular, otherwise multiplying the left with $\lambda^{-2}\mathbf{C}^{-1}$ we would get the quadratic pencils

$$\mathbf{Q_2}(\lambda) \coloneqq \lambda^{-2}\mathbf{I} + \mathbf{B}_1\lambda^{-1} + \mathbf{C}_1,\ \mathbf{B}_1 := \mathbf{C}^{-1}\mathbf{B},\ \ \mathbf{C}_1 := \mathbf{C}^{-1}\mathbf{A}. \tag{14}$$

Let $\mathrm{Rank}(\mathbf{A}) = n - p,\ \mathrm{Rank}(\mathbf{B}) = n - q,\ \mathrm{Rank}(\mathbf{C}) = n - r$ where $0 < p < n,\ 0 \le q \le n$ and $0 < r < n$ and let $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}, \{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$ orthonormal base of vector space $\mathbb{C}^n$ consisting of eigenvectors of the corresponding matrix **A, B** and **C** respectively. Without loss of generality, we take

$$\mathbf{A}\mathbf{y}_i = 0, \quad i = 1,2,\cdots,p, \quad \mathbf{B}\mathbf{z}_j = 0, j = 1,2,\cdots,q, \quad \mathbf{C}\mathbf{w}_k = 0\,, \quad k = 1,2,\cdots,r. \tag{15}$$

Now we state and prove theorems that provide localization of the parameters μ and ξ.

Theorem 2 Let $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$ and $\xi < \mu$ then stands $\mu > a_1 > c_1 > \xi$ where

$$a_1 := max\left\{-\frac{\mathbf{y}^H\mathbf{C}\mathbf{y}}{\mathbf{y}^H\mathbf{B}\mathbf{y}} : \mathbf{y} \in \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p\}\right\} \text{ and } c_1 := min\left\{-\frac{\mathbf{y}^H\mathbf{C}\mathbf{y}}{\mathbf{y}^H\mathbf{B}\mathbf{y}} : \mathbf{y} \in \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p\}\right\}.$$

Proof

Let $\mathbf{y} \in \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p\}$. From this follows

$$\mu\,\mathbf{y}^H\mathbf{B}\mathbf{y} + \mathbf{y}^H\mathbf{C}\mathbf{y} > 0 \tag{16}$$

$$\xi\,\mathbf{y}^H\mathbf{B}\mathbf{y} + \mathbf{y}^H\mathbf{C}\mathbf{y} < 0 \tag{17}$$

Subtracting the second from the first inequality we get

$$(\mu - \xi)\mathbf{y}^H\mathbf{B}\mathbf{y} > 0 \overset{\xi<\mu}{\Longrightarrow} \mathbf{y}^H\mathbf{B}\mathbf{y} > 0\,. \tag{18}$$

From (16) and (18) we get

$$\mu > -\frac{\mathbf{y}^H\mathbf{C}\mathbf{y}}{\mathbf{y}^H\mathbf{B}\mathbf{y}} \text{ for every } \mathbf{y} \in \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p\}.$$

Latest means

$$\mu > a_1 = \max\left\{-\frac{\mathbf{y}^H\mathbf{C}\mathbf{y}}{\mathbf{y}^H\mathbf{B}\mathbf{y}} : \mathbf{y} \in \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p\}\right\}$$

Analogously, we prove

$$\xi < c_1 = \min\left\{-\frac{\mathbf{y}^H\mathbf{C}\mathbf{y}}{\mathbf{y}^H\mathbf{B}\mathbf{y}} : \mathbf{y} \in \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p\}\right\}. \blacksquare$$

Theorem 3 Let $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$ and $\xi < \mu$ then stands $\mu > a_2$ and $c_2 > \xi$ where

$$a_2 \coloneqq \max\left\{-\frac{\mathbf{w}^H\mathbf{B}\mathbf{w}}{2\mathbf{w}^H\mathbf{A}\mathbf{w}} : \mathbf{w} \in \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r\} \text{ and } \mathbf{w}^H\mathbf{A}\mathbf{w} > 0\right\}$$

and

$$c_2 := \min\left\{-\frac{\mathbf{w}^H\mathbf{B}\mathbf{w}}{2\mathbf{w}^H\mathbf{A}\mathbf{w}} : \mathbf{w} \in \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r\} \text{ and } \mathbf{w}^H\mathbf{A}\mathbf{w} < 0\right\}$$

Proof
Let $\mathbf{w} \in \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r\}$. From this follows

$$\mu^2\mathbf{w}^H\mathbf{A}\mathbf{w} + \mu\mathbf{w}^H\mathbf{B}\mathbf{w} > 0 \quad (19)$$

$$\xi^2\mathbf{w}^H\boldsymbol{A}\boldsymbol{w} + \xi\mathbf{w}^H\mathbf{B}\mathbf{w} < 0 \quad (20)$$

Subtracting the inequality (20) from inequality (19) yields

$$(\mu - \xi)(\mu + \xi)\mathbf{w}^H\mathbf{A}\mathbf{w} > -(\mu - \xi)\mathbf{w}^H\mathbf{B}\mathbf{w} \overset{\xi<\mu}{\Longrightarrow} (\mu + \xi)\,\mathbf{w}^H\mathbf{A}\mathbf{w} > -\mathbf{w}^H\mathbf{B}\mathbf{w} \quad (21)$$

From (21) we get

$$\mu + \xi > -\frac{\mathbf{w}^H\mathbf{B}\mathbf{w}}{\mathbf{w}^H\mathbf{A}\mathbf{w}} \quad \text{for every} \quad \mathbf{w} \in \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\} \quad \text{for which} \quad \mathbf{w}^H\mathbf{A}\mathbf{w} > 0. \quad (22)$$

From (22) and $\mu - \xi > 0$ follows

$$\mu > a_2 = \max\left\{-\frac{\mathbf{w}^H\mathbf{B}\mathbf{w}}{2\mathbf{w}^H\mathbf{A}\mathbf{w}} : \mathbf{w} \in \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r\} \text{ and } \mathbf{w}^H\mathbf{A}\mathbf{w} > 0\right\}.$$

Analogously we prove that

$$\xi < c_2 = \min\left\{-\frac{\mathbf{w}^H\mathbf{B}\mathbf{w}}{2\mathbf{w}^H\mathbf{A}\mathbf{w}} : \mathbf{w} \in \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r\} \text{ and } \mathbf{w}^H\mathbf{A}\mathbf{w} < 0\right\}. \blacksquare$$

These two theorems can be combined into one, which reads:

Theorem 4 Let $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$, $\xi < \mu$ and $a := \max(a_1, a_2)$ and $c := \min(c_1, c_2)$ then applies $\mu > a > c > \xi$ where a_1, a_2, c_1 and c_2 defined as in Theorem 2 and Theorem 3.

If we now apply Algorithm 1 to determine the parameter μ for the initial vector $\mathbf{x}_0$ we take the vector through which we specify a. Analogously if we apply Algorithm 1 (Fig. 2) for determining parameter ξ for initial vector $\mathbf{x}_0$ we take the vector through which we have determined c.

There remains to consider what happens in the case of the singularity of the matrix **B**.

Theorem 5 Let $\mathbf{Q}(\xi) < 0 < \mathbf{Q}(\mu)$, $\xi < \mu$, $\text{Rank}(\mathbf{B}) = n - q$ and $q > 0$ than $\mu > 0$ or $\xi < 0$.

Proof
Let $\mathbf{z} \in \{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_q\}$. From this follows

$$\mu^2\mathbf{z}^H\mathbf{A}\mathbf{z} + \mathbf{z}^H\mathbf{C}\mathbf{z} > 0 \quad (23)$$

$$\xi^2\mathbf{z}^H\mathbf{A}\mathbf{z} + \mathbf{z}^H\mathbf{B}\mathbf{z} < 0 \quad (24)$$

Subtracting inequality (24) from inequality (23) yields

$$(\mu-\xi)(\mu+\xi)\mathbf{z}^H\mathbf{A}\mathbf{z} > 0 \overset{\xi<\mu}{\Longrightarrow} (\mu+\xi)\mathbf{z}^H\mathbf{A}\mathbf{z} > 0 \quad (25)$$

If $\mathbf{z}^H\mathbf{A}\mathbf{z} > 0$ than $\mu+\xi > 0$ respectively $\mu > 0$. Analogously from $\mathbf{z}^H\mathbf{A}\mathbf{z} < 0$ follows $\xi < 0$.■

Remark: From inequality (25) it is clear that $\mathbf{z}^H\mathbf{A}\mathbf{z}$ has the same sign for all $\mathbf{z} \in \{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_q\}$. So it is enough to check the sign of $\mathbf{z}_1^H\mathbf{A}\mathbf{z}_1$ and for appropriate vector take $\mathbf{z}_1$ and in the case of singularities of the matrix **B** in order to reduce the costs of determining the initial vector $\mathbf{x}_0$ it is taken $\mathbf{x}_0=\mathbf{z}_1$.

Conclusion

These theoretical foundations enable a better choice of the initial vector $\mathbf{x}_0$ and thus accelerate the Algorithm 1 (Fig. 2). In these theoretical foundations we have used some properties of the matrix **A**, **B** and **C.** Those properties are for example rank of the matrix, eigenvalues and eigenvectors of the already mentioned matrix.

Further research will go in the direction of use of features of the matrix **A**, **B** and **C**, which determine square pencil that is used for improvement of localisation of the parameters ξ and μ.

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