Annals of DAAAM for 2011 & Proceedings of the 22nd International DAAAM Symposium, Volume 22, No. 1, ISSN 1726-9679
ISBN 978-3-901509-83-4, Editor B. Katalinic, Published by DAAAM International, Vienna, Austria, EU, 2011
Make Harmony between Technology and Nature, and Your Mind will Fly Free as a Bird
Annals & Proceedings of DAAAM International 2011

ON AN APPLICATION OF NEIGHBORHOOD FOR A CLASS OF HARMONIC UNIVALENT FUNCTIONS

CATAS, A[driana] & DUBAU, C[alin]

Abstract: *In the present paper we investigate a class of harmonic univalent functions in the open unit disk* $U = \{z \in C : |z| < 1\}$ *under certain conditions involving a differential operator. We will give an application of neighborhood in this sens.*

Key words: *harmonic univalent function, minimal surfaces distortion bounds, neighborhood.*

1. INTRODUCTION

Harmonic mappings in the plane are univalent complex-valued harmonic functions whose real and imaginary parts are not necessarily conjugate. In other words, the Cauchy-Riemann equations need not to be satisfied, so the functions need not to be analytic (Duren, 2004).

In studying harmonic mappings of simply connected domains in the plane, there is no essential loss of generality in taking the unit disk as the domain of definition.

Although harmonic mappings are natural generalizations of conformal mappings, they were studied originally because of their natural role in parametrizing minimal surfaces.

In two papers [Chuaqui et. al., 2003] were introduced a notion of Schwarzian derivative for a locally univalent harmonic mapping and showed that it retains some of the classical properties of the Schwarzian of an analytic function. In these investigations it was fruitful to identify the harmonic mapping with its local lift to a minimal surface.

2. MATERIAL AND METHOD

For a continuous complex-valued function $f=u+iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simple connected domain we can write $f=h+g$, where h and g are analytic in D. A necessary and sufficient condition for f to be univalent and sense preserving in D is that $|h'(z)|>|g'(z)|$, $Z \in D$. See (Clunie & Sheil-Small, 1984) for more details.

Denote by $S_H(n)$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disc U for which $f(0)=f_z(0)-1=0$. Then for $f = h + \overline{g} \in S_H(n)$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad g(z) = \sum_{k=n}^{\infty} b_k z^k \tag{1}$$

Let $ST_H(n,m)$ denote the family of functions $f = h + \overline{g}_m$ that are harmonic in D with the normalization

$$h(z) = z - \sum_{k=n+1}^{\infty} |a_k| z^k, \quad g_m(z) = (-1)^m \sum_{k=n}^{\infty} |b_k| z^k \tag{2}$$

Using the following differential operator

$$I^m h(z) = (l+1)^m z + \sum_{k=n+1}^{\infty} [1+\lambda(k-1)+l]^m C(\delta,k) a_k z^k$$

where

$$C(\delta,k) = \binom{k+\delta-1}{\delta} = \frac{\Gamma(k+\delta)}{\Gamma(k)\Gamma(1+\delta)}, \tag{3}$$

we will obtain the new results. By making use of the class $S_H(n)$ where f belonging to, we can introduce the relation

$$I^m f(z) = I^m h(z) + (-1)^m \overline{I^m g(z)}. \tag{4}$$

where

$$I^m g(z) = (l+1)^m z + \sum_{k=n}^{\infty} [1+\lambda(k-1)+l]^m C(\delta,k) b_k z^k \tag{5}$$

A function $f \in S_H(n)$ is said to be in the class $AL_H(m,\delta,\alpha,\lambda,l)$ if

$$\mathrm{Re}\left\{\frac{I^{m+1} f(z)}{I^m f(z)}\right\} \geq \alpha,\ 0 \leq \alpha \leq 1. \tag{6}$$

This class includes a variety of well-known subclasses of $S_H(n)$. For example we can reobtain several classes introduced earlier by (Jahangiri, 1998), (Shaqsi & Darus, 2008), (Ahuja & Jahangiri, 2003). Further we will give an application of the class mentioned above.

Finally, using the subclass

$$ALT_H(m,\delta,\alpha,\lambda,l) \equiv AL_H(m,\delta,\alpha,\lambda,l) \cap ST_H(n,m), \tag{7}$$

we can achieve the next section.

3. RESULTS AND DISSCUSIONS

For this section we will define a generalized (n,η)-neighborhood of a function f given in (2) to be the set

$$N_{n,\eta}(f) = \{F_m(z) \in ST_H :$$

$$\sum_{k=n+1}^{\infty} \frac{\left[1-\alpha+\lambda(k-1)+l\right] d_k(m,\lambda,l) C(\delta,k)}{(l+1)^m (1-\alpha+l)} \left|a_k - A_k\right| +$$

$$+\sum_{k=n}^{\infty}\frac{\left[1+\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}{(l+1)^m(1-\alpha+l)}\left|b_k-B_k\right|\le\eta\}$$

where

$$d_k(m,\lambda,l)=[1+\lambda(k-1)+l]^m$$

and

$$F_m(z)=z-\sum_{k=n+1}^{\infty}\left|A_k\right|z^k+(-1)^m\sum_{k=n}^{\infty}\left|B_k\right|\bar{z}^{-k}. \qquad (8)$$

Now we can state the following theorem.

Let $f_m=h+\overline{g}_m$ be given by (2). If the functions f_m satisfy the conditions

$$\sum_{k=n+1}^{\infty}k\frac{\left[1-\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}{(l+1)^m(1-\alpha+l)}\left|a_k\right|+$$
$$+\sum_{k=n}^{\infty}k\frac{\left[1+\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}{(l+1)^m(1-\alpha+l)}\left|b_k\right|\le \qquad (9)$$
$$\le 1-U(m,\lambda,l)$$

denoted by (9)
and

$$\eta\le\frac{n-\alpha}{1+n-\alpha}\left(1-U(m,\lambda,l)\right) \qquad (10)$$

where

$$U(m,\lambda,l)=\frac{\left[1+\alpha+\lambda(k-1)+l\right]d_n(m,\lambda,l)C(\delta,n)}{(l+1)^m(1-\alpha+l)}\left|b_n\right|$$

then $N_{n,\eta}(f)\subset ALT_H(m,\delta,\alpha,\lambda,l)$.

Indeed, knowing that f_m satisfy the condition (8) and $F_m\in N_{n,\eta}(f)$ one obtains

$$\sum_{k=n+1}^{\infty}\frac{\left[1-\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}{(l+1)^m(1-\alpha+l)}\left|A_k\right|+$$
$$+\sum_{k=n}^{\infty}\frac{\left[1+\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}{(l+1)^m(1-\alpha+l)}\left|B_k\right|\le$$

$$\le\eta+\sum_{k=n+1}^{\infty}\left(\frac{\left[1-\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}{(l+1)^m(1-\alpha+l)}\left|a_k\right|+\right.$$
$$\left.+\frac{\left[1+\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}{(l+1)^m(1-\alpha+l)}\left|b_k\right|\right)+U(m,\lambda,l)\le$$

$$\eta+\frac{1}{1+n-\alpha}\sum_{k=n+1}^{\infty}k\left(\frac{\left[1-\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}{(l+1)^m(1-\alpha+l)}\left|a_k\right|+\right.$$
$$\left.\frac{\left[1+\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}{(l+1)^m(1-\alpha+l)}\left|b_k\right|\right)+U(m,\lambda,l)\le$$
$$\eta+\frac{1}{1+n-\alpha}(1-U(m,\lambda,l))+U(m,\lambda,l)\le 1.$$

Hence, for η satisfying inequality (9) we deduce that $f_m\in ALT_H(m,\delta,\alpha,\lambda,l)$.

4. CONCLUSION

Using the model above, we can also have a representation of the functions in the class $ALT_H(m,\delta,\alpha,\lambda,l)$ from which we also establish the extreme points of closed hulls of the class.

So, a function belonging to this class can be expressed as

$$f_m(z)=Xh(z)+\sum_{k=n+1}^{\infty}X_k h_k(z)+\sum_{k=n}^{\infty}Y_k g_{m_k}(z)$$

where

$$h(z)=z$$

and

$$h_k(z)=z-\frac{(l+1)^m(1-\alpha+l)}{\left[1-\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}z^k,$$
$$k=n+1,\ n+2,\ \ldots$$

and

$$g_{m_k}(z)=z+(-1)^m\frac{(l+1)^m(1-\alpha+l)}{\left[1+\alpha+\lambda(k-1)+l\right]d_k(m,\lambda,l)C(\delta,k)}\bar{z}^{-k}$$

$$k=n,\ n+1,\ \ldots$$

with

$$X_k\ge 0, Y_k\ge 0, Y_n\ge 0, X\ge 0, X=1-\sum_{k=n+1}^{\infty}X_k+\sum_{k=n}^{\infty}Y_k.$$

In particular, the extreme points of the class $ALT_H(m,\delta,\alpha,\lambda,l)$ are $\left\{h_k\right\}$ and $\left\{g_{m_k}\right\}$.

5. REFERENCES

Al-Shaqsi, K. & Darus, M. (2008). On Harmonic Functions Defined by Derivative Operator. *Journal of Inequalities and Applications*, Vol. 2008, Article ID 263413, doi: 10.1155/2008/263413, ISSN 1025-5834

Chuaqui, M.; Duren, P. & Osgood, B. (2003). The Schwarzian derivative for harmonic mappings, *J. Analyse Math.* 91 (2003), 329-351.

Chuaqui, M.; Duren, P. & Osgood, B. Ellipses, near ellipses, and harmonic Mobius transformations, *Proc. Amer. Math. Soc.*, to appear.

Clunie, J. & Sheil-Small, T. (1984). Harmonic Univalent Functions. *Ann. Acad. Sci. Fenn*, Ser. A I. Math. 9(1984), pp. 3-25, ISSN 0066-1953

Duren, P. (2004). *Harmonic mappings in the plane*, Cambridge University Press, ISBN 0-521-64121-7, United Kigdom