Annals of DAAAM for 2011 & Proceedings of the 22nd International DAAAM Symposium, Volume 22, No. 1, ISSN 1726-9679
ISBN 978-3-901509-83-4, Editor B. Katalinic, Published by DAAAM International, Vienna, Austria, EU, 2011
Make Harmony between Technology and Nature, and Your Mind will Fly Free as a Bird
Annals & Proceedings of DAAAM International 2011

SENSITIVITY ANALYSIS FOR LQ CONTROLLER IN THE POSITION AND THE INCREMENTAL FORM

BURLAK, V[ladimir]; PIVONKA, P[etr] & DOKOUPIL, J[akub]

Abstract: *This article is focused on the comparing of the sensitivity of LQ controller in the position and the incremental form. The sensitivity analysis was used to compare sensitivity of LQ controllers. Efficient solving of the sensitivity function was deduced here. Two criterial functions were used for the sensitivity analysis*

Key words: *sensitivity analysis, sensitivity function, optimal control*

1. INTRODUCTION

Linear Quadratic optimal controller (LQ) can be used in the positional and the incremental form. The positional form is used for the setting of LQ controllers in (Lin, 2007) and (Sinha, 2007). The incremental form is used in the theory of predictive control. This form can be found (Bao-Cang, 2010). Are both forms equivalent in the sensitivity to changes in parameters? Which form is more robust? The answers to these questions are searched in this article. The sensitivity analysis was used to verify the sensitivity of both forms. The sensitivity function must be known for the using sensitivity analysis. The efficient solving of the sensitivity function is deduced here. Two criterial functions were used for sensitivity analysis. First criterial function is used for LQ controller settings. Second criterial function is the sum of squared deviations of output from the reference value. The ground of sensitivity analysis can be found in (Saltelli, 2008).

2. STATE SPACE REPREZENTATION OF THE SYSTEM AND CONTROL LAW

Pseudo state space representation is used for discrete dynamic systems. The state space vector is composed from the delayed values of input and output control system. This model is very advantageous because all state variables are measurable. An equation (1) and (2) show the relationship between the parameter vector and the discrete transfer function. The detailed description of the pseudo-state representation can be found in (Bohm, 2011).

$$F_D(z^{-1}) = \frac{b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \tag{1}$$

$$\begin{aligned}\theta^T &= [b_1 \;\; \dots \;\; b_m \;\; -a_1 \;\; \dots \;\; -a_n] = \\ &= [\theta_1 \;\; \dots \;\; \theta_{m+n}]\end{aligned} \tag{2}$$

The state equations have for position controller the form (3)

$$\begin{aligned}x(k+1,\theta) &= A(\theta)x(k,\theta) + B(\theta)u(k) \\ y(k,\theta) &= C(\theta)x(k,\theta) + D(\theta)u(k)\end{aligned} \tag{3}$$

where $x(k,\theta)$ is the vector of the state variables in step k, $u(k)$ is system input, $y(k,\theta)$ is system output and θ is the vector of parameters in the form (2).

The equations (4) are state equations for incremental controller

$$\begin{aligned}\begin{bmatrix} x(k+1,\theta) \\ u(k+1) \end{bmatrix} &= \begin{bmatrix} A(\theta) & B(\theta) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(k,\theta) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u(k) \\ y(k,\theta) &= [C(\theta) \quad D(\theta)] \begin{bmatrix} x(k,\theta) \\ u(k) \end{bmatrix}\end{aligned} \tag{4}$$

where $\Delta u(k) = u(k+1) - u(k)$.

If the matrix $D(\theta)$ is zero matrix, then we can rewrite the equations (3) or (4) to equations in the form (5).

$$\begin{aligned}x_q(k+1,\theta) &= A_q(\theta)x_q(k,\theta) + B_q u_q(k) \\ y(k,\theta) &= C_q(\theta)x_q(k,\theta)\end{aligned} \tag{5}$$

The criterial functions (6) and (8) are used for the sensitivity analysis in this paper. A control law (7) is given by minimizing the first criterial function in the form (6). A finding a matrix K is described for example in (Sinha, 2007) or (Lin, 2007).

$$J_1 = \sum_{i=0}^{\infty} x_q^T(i) Q x_q(i) + u_q^T(i) R u_q(i) \tag{6}$$

$$u_q(k) = -K x_q(k) \tag{7}$$

Second criterial function (9) is the sum of squared deviations of output from the reference value. A generator reference trajectory is included in state equations (5) in this paper. The matrix T is such a constant matrix in order to fulfill the relationship (9). The variable $w(k)$ represents the reference value.

$$J_2 = \sum_{i=0}^{\infty} e^T(i)\, e(i) = \sum_{i=0}^{\infty} x_q^T(i) N^T(\theta) N(\theta) x_q(i) \tag{8}$$

$$\begin{aligned}e(k) &= \big(w(k) - y(k)\big) = \big(T - C(\theta)\big)x_q(k) = \\ &= N(\theta)x_q(k)\end{aligned} \tag{9}$$

3. SENSITIVITY FUNCTIONS

The sensitivity analysis was used to the comparison of LQ controllers in the position and the incremental form. It was necessary to deduce the sensitivity function. This function defines the relationship between the change of parameters θ and the value of criterial function $J(\theta)$ in the form (6) or (8). The derivation assumes the following: the parameters are constant during control (constant deviations), the controllers are fixed. The approximation of the sensitivity function can be defined by (10)

$$\Delta J(\theta) \approx \sum_{j=1}^{m+n} \frac{\partial J(\theta)}{\partial \theta_j} \Delta\theta_j \tag{10}$$

where $\Delta J(\theta)$ is deviation of the value of criterial function and $\Delta\theta_j$ is deviation of the j-th parameter. Furthermore, we use an autonomous dynamic system with the first state equation in the form (11)

$$\begin{bmatrix} x_q(k,\theta) \\ \dfrac{\partial x_q(k,\theta)}{\partial \theta_j} \end{bmatrix} =$$
$$= \begin{bmatrix} A_q(\theta) - B_q K & 0 \\ \dfrac{\partial A(\theta)}{\partial \theta_j} & A_q(\theta) - B_q K \end{bmatrix}^k \begin{bmatrix} x_q(0) \\ \dfrac{\partial x_q(0)}{\partial \theta_j} \end{bmatrix} =$$
$$= E^k(\theta) x_l(0,\theta) = x_l(k,\theta) \quad (11)$$

with initial condition (12).

$$x_l(0) = \begin{bmatrix} x_q(0) \\ 0 \end{bmatrix} \quad (12)$$

Now we can write the partial derivative of the criterial function according to the j-th parameter in the form (13)

$$\frac{\partial J(\theta)}{\partial \theta_j} = x_l^T(0) \sum_{i=0}^{\infty} \left[\left(E^i(\theta) \right)^T S E^i(\theta) \right] x_l(0) =$$
$$= S + E^T(\theta) P_S(\theta) E(\theta) \quad (13)$$

Where

$$P_S(\theta) = S + E^T(\theta) P_S(\theta) E(\theta). \quad (14)$$

The equation is valid for criterial functions (6) and (8). The matrix S is different only. The equation (15) is valid for the criterial function in the form (6) and the equation (16) is valid for the criterial function in the form (8).

$$S = S_1 = 2 \begin{bmatrix} 0 \\ I \end{bmatrix} (Q + K^T R K) [I \quad 0] \quad (15)$$

$$S = S_{2,j}(\theta) = 2 \begin{bmatrix} 0 \\ I \end{bmatrix} N(\theta) [I \quad 0] + \begin{bmatrix} I \\ 0 \end{bmatrix} \frac{\partial N(\theta)}{\partial \theta_j} [I \quad 0] \quad (16)$$

I is the identity matrix and 0 is the zero matrix. Both matrices are square. Their dimensions are equal as dimension matrix $A_q(\theta)$.

4. SIMULATION RESULTS

A transfer function of the system was assumed in the form (17).

$$F(s) = \frac{K_s}{(T_1 s + 1)(T_2 s + 1)} \quad (17)$$

Every simulation consisted of the following:

The parameters T_1, T_2 and gain K_S were set to $T_1 = 10s$, $T_2 = 50s$ and $K_S = 20$. The system was discretized. Controller parameters were set for this system. Controller parameters were set to this system and the value criterial function was calculated by (6). Furthermore, values have been changed, the system was discretized and the value of the deviation parameter was calculated. Further, the deviation criterial function was calculated by (10). In conclusion, changes in the relative value of the criterial function were calculated according to equation (18).

$$\delta = 100 \frac{|\Delta J(\theta)|}{J(\hat{\theta})} \quad (18)$$

where $\hat{\theta}$ is the vector of parameter, to which the controller was designed. The sampling period was set to 0.1s. The parameter R was set to one. Matrix Q was given by the equation (19)

$$Q = q\left(T - C_q\right)^T \left(T - C_q\right) + I \quad (19)$$

where a coefficient q was equal to 10 and I is identity matrix. Vector (19) was chosen as an initial condition for the LQ controller in position form. The vector (20) was chosen as an initial condition for the LQ controller in incremental form. Figure 1 contains the results for criterial function in the form (6) and Figure 2 contains the results for criterial function in the form (8). The equivalence of both approaches is shown in Figure 1 and Figure 2. This equivalence is also evident from the equation (11) and the form of a matrix pseudo-state representation. Pseudo-state representation can be found for example in (Bohm, 2011).

$$x_q(0) = [0.1 \quad 0.1 \quad 0 \quad 0 \quad 1\,]^{\mathrm{T}} \quad (19)$$

$$x_q(0) = [0.1 \quad 0.1 \quad 0 \quad 0 \quad 1 \quad 0\,]^{\mathrm{T}} \quad (20)$$

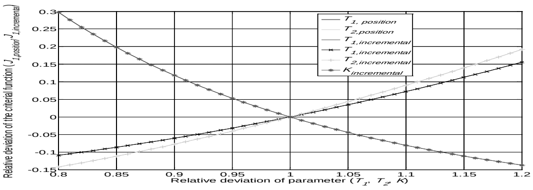

Fig. 1. Simulation results for criterial function J_1

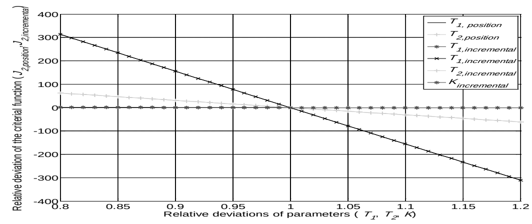

Fig. 2. Simulation results for criterial function J_2

5. CONCLUSION

This article dealt with the comparison of LQ controllers in the position and the incremental form. The sensitivity analysis was used to compare the sensitivity of both controllers to change the parameters of the plant. Two criterial functions were used to check sensitivity to changes in parameters of the system. Derived sensitivity functions were verified in the simulation of the same system. Simulation results show the same sensitivity to changes in parameters of controlled system with identical rates in the criterial functions (6) and (8).

6. ACKNOWLEDGEMENTS

This work was supported by grant "Research of Modern Methods and Approaches in Automation" from the Internal Grant Agency of Brno University of Technology (grant No. FEKT-S-11-6) and Grant Agency of the Czech Republic (102/09/H081 SYNERGY – Mobile Sensoric Systems and Network).

7. REFERENCES

Bao-Cang, D. (2010). *Modern predictive control,* CRC Press, ISBN 978-1-4200-8530-3, USA

Bohm, J. (2011). Adaptive LQ controllers, *Available from:* http://crr.vutbr.cz/system/files/brozura_07_1104.pdf *Accessed:* 2011-06-15

Lin, F. (2007). *Robust control,* John Wiley & Sons, ISBN 978-0-470-03191-9, England

Sinha, A. (2007). *Linear systems, optimal and robust control,* CRC Press, ISBN 0-8493-9317-9, New York

Saltelli, A. et al. (2008). *Global sensitivity analysis: The primer,* John Wiley & Sons, ISBN 978-0-470-05997-5, England