Annals of DAAAM for 2011 & Proceedings of the 22nd International DAAAM Symposium, Volume 22, No. 1, ISSN 1726-9679
ISBN 978-3-901509-83-4, Editor B. Katalinic, Published by DAAAM International, Vienna, Austria, EU, 2011
Make Harmony between Technology and Nature, and Your Mind will Fly Free as a Bird
Annals & Proceedings of DAAAM International 2011

A FINITE VOLUME METHOD FOR A GEOMECHANICS PROBLEM

BIJELONJA, I[zet]

Abstract: *This paper present development of a finite volume based method for modeling of elasto-plastic deformation of solids obeying the Drucker-Prager yield criterion. The numerical method is based on the solution of the integral form of conservation equations governing momentum balance. To solve resulting set of coupled linear algebraic equations a segregated approach is employed. The method is applicable to the meshes consisting polyhedral finite volume cells. Numerical results show very good accuracy of the method.*

Key words: *numerical analysis, finite volume method, geomechanics, elasto-plastic deformation*

1. INTRODUCTION

The finite volume (FV) method has so far been applied to a wide range of problems of solid mechanics. A series of the FV applications to the elasto-plastic solids has started by the work presented in (Demirdzic & Martinovic, 1993). All of these works are in conjunction with the von-Mises yield criterion. The finite volume method for analysis of elasto-plastic deformation in geomechanics has not applied yet.

In this paper a finite volume based formulation for elasto-plastic deformation of solids associated with the Drucker-Prager yield criterion is presented. The method is based on the discretisation of the integral form of momentum balance equation. A collocated variable arrangement is used, and by a segregated procedure resulting set of coupled algebraic equations is solved. The numerical discretisation used in this paper is developed by employing the finite-volume method procedures described in detail (Demirdzic & Muzaferija, 1994; Demirdzic & Muzaferija, 1995).

The method is applied to a case of a strip footing on the soil stratum problem. The FV method calculations are compared with the finite element and boundary element solutions.

In the next section the governing equations and constitutive relations are given. Then, a brief description of the applied finite volume discretisation procedure is outlined. The method's capabilities are demonstrated by applying it to a study case.

2. MATHEMATICAL FORMULATION AND NUMERICAL DISCRETIZATION

In this section, governing and constitutive equations that describe elasto-plastic deformation of solids, as well as discretization procedures in the context of FV formulation are briefly outlined.

The law of conservation of linear momentum in an integral form for a body of volume V, bounded by the surfaces with outward pointing surface vector $\mathbf{s}$, is

$$\frac{\partial}{\partial t}\int_V \rho \frac{\partial \mathbf{u}}{\partial t}\mathrm{d}V = \int_S \boldsymbol{\sigma}\cdot \mathrm{d}\mathbf{s} + \int_V \rho \mathbf{f}_\mathrm{b}\mathrm{d}V, \tag{1}$$

where ρ is mass density, $\mathbf{u}$ is displacement vector, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, and $\mathbf{f}_\mathrm{b}$ is the resultant body force.

It is assumed here that angular momentum balance equations are satisfied identically due the shear stresses conjugate principle, i.e.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^\mathrm{T}. \tag{2}$$

Within the context of associated plasticity theory, assuming the Drucker-Prager yield criterion, following incremental constitutive equations describe behavior of a body in a case of elasto-plastic deformation:

$$\delta\boldsymbol{\sigma} = \mu\left[\mathrm{grad}\,\delta\mathbf{u} + (\mathrm{grad}\,\delta\mathbf{u})^\mathrm{T}\right] + \lambda(\mathrm{div}\,\delta\mathbf{u})\mathbf{I} - \beta\,\frac{2\mu\left(\alpha'\mathbf{I} + \frac{\sqrt{3}}{2}\frac{\boldsymbol{\sigma}'}{\sigma_e}\right) + 3\lambda\alpha'\mathbf{I}}{\mu + 9\alpha'^2 K' + H'} \left(\sqrt{3}\frac{\mu}{2\sigma_e}\boldsymbol{\sigma}' : \left[\mathrm{grad}\,\delta\mathbf{u} + (\mathrm{grad}\,\delta\mathbf{u})^\mathrm{T}\right] + 3\alpha' K'(\mathrm{div}\,\delta\mathbf{u})\mathbf{I}\right), \tag{3}$$

where $\boldsymbol{\varepsilon}$ is the linear strain tensor, $\boldsymbol{\sigma}'$ is the deviatoric Cauchy stress tensor, $\sigma_e = (3/2\boldsymbol{\sigma}' : \boldsymbol{\sigma}')^{1/2}$ is the effective stress, $\mathbf{I}$ is the identity tensor, λ and μ are the Lamé constants related to the Young's modulus E and the Poisson's ratio ν, and H' is strain hardening parameter. Constants α' and K' in Eq. (3) are associated with the Drucker-Prager yield criteria:

$$\alpha'\,\mathrm{tr}\,\boldsymbol{\sigma} + \sigma_e/\sqrt{3} - K' = 0, \tag{4}$$

and related to the internal friction of the deforming body φ and cohesion of the material c' as follows:

$$\alpha' = \frac{2\sin\varphi'}{\sqrt{3}(3-\sin\varphi')}, \quad K' = \frac{6c'\cos\varphi'}{\sqrt{3}(3-\sin\varphi')}. \tag{5}$$

Parameter β in Eq. (3) equals one in a case of elasto-plastic deformation, and equals zero in a case of a elastic deformation reducing Eq. (3) to the well known Hooke's constitutive law of an ideal elastic body.

Equations (1) and (3) make a close set of three equations with three unknown functions δu_i of spatial coordinates and time. To complete the mathematical model, initial and boundary conditions must be specified. As initial conditions, the displacements, and in transient cases velocities, have to be

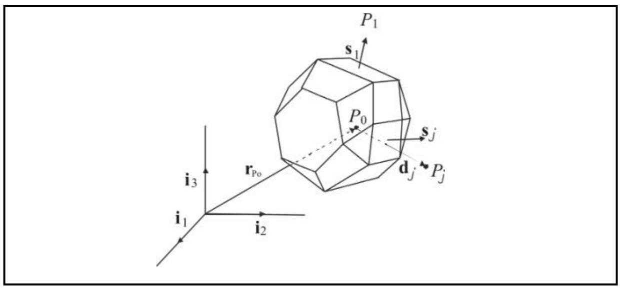

Fig. 1. A control volume of an arbitrary shape

specified at all points of the solution domain. Boundary conditions have to be specified at all times at all solution domain boundaries.

Governing equations are discretised by employing the finite volume procedures described in detail in (Bijelonja, 2002) which is adopted from (Demirdžić & Muzaferija, 1994; Demirdžić & Muzaferija, 1995). In order to obtain discrete counterpart of Eq. (1), the time interval of interest is divided into an arbitrary number of time steps δt, and the space is discretised by a number of contiguous, non-overlapping control volumes, with computational points at their centres, Fig. 1.

Introducing constitutive relations (3) into Eq. (1), the momentum equation is written for each control volume as follows:

$$\frac{\partial}{\partial t}\int_V \rho \frac{\partial \delta \mathbf{u}}{\partial t}\, dV = \sum_{j=1}^{n}\int_S \mu \left(\text{grad } \delta\mathbf{u}\right)\cdot d\mathbf{s} + \sum_{j=1}^{n}\int_S Q_{\phi a}\cdot d\mathbf{s} + \int_V \rho \mathbf{f}_b\, dV \qquad (6)$$

$$Q_{\phi a} = \mu\left(\text{grad } \delta\mathbf{u}\right)^{\mathrm{T}} + \lambda\left(\text{div } \delta\mathbf{u}\right)\mathbf{I} - \beta \frac{2\mu\left(\alpha'\mathbf{I} + \frac{\sqrt{3}}{2}\frac{\boldsymbol{\sigma}'}{\sigma_e}\right) + 3\lambda\alpha'\mathbf{I}}{\mu + 9\alpha'^2 K' + H'} \left(\sqrt{3}\frac{\mu}{2\sigma_e}\boldsymbol{\sigma}' : \left[\text{grad } \delta\mathbf{u} + \left(\text{grad } \delta\mathbf{u}\right)^{\mathrm{T}}\right] + 3\alpha' K'\left(\text{div } \delta\mathbf{u}\right)\mathbf{I}\right),$$

where n is the number of cells which share cell-faces with the cell whose balance is considered.

In order to evaluate integrals in the above equation the mid-point formula is used and a linear variation of displacement increment components is assumed. By integrating equation (6) over a time interval δt (time discretisation scheme is not discussed here) coupled sets of algebraic equations with three unknown δu_i are obtained. The resulting algebraic equations are solved by use of a segregated solution algorithm by temporarily decoupling of the system of equations.

After the solution of the set of algebraic equations, before advance to the next load increment the total displacement vector and the total Cauchy stress tensor are updated at each control volumes centres.

3. STUDY CASE

An alasto-plastic analysis of flexible strip footing under uniform pressure is analysed (Fig. 2.). A plane strain deformation is assumed. The soil stratum is considered ideally plastic material obeying the associated Mohr-Coulomb (M-C) yield criterion with material properties given in Fig. 2. The Mohr-Coulomb criterion can be simulated using Drucker-Prager yield criterion substituting α' and K' in Eq.(5) as follows:

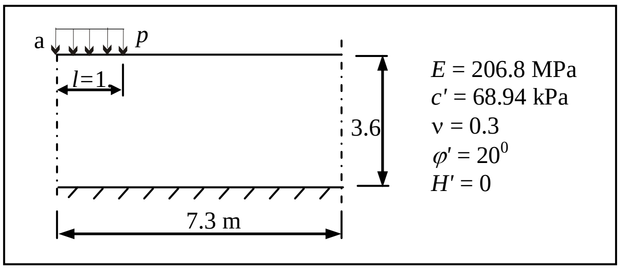

Fig. 2. Geometry and material properties for the strip footing problem

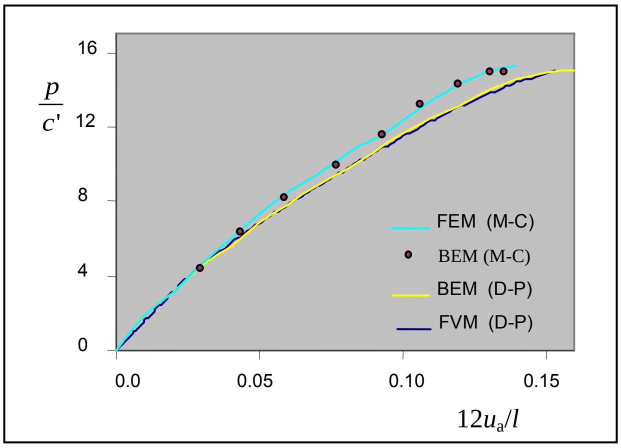

Fig. 3. Load-displacement curves for the strip footing problem

$$\alpha' = \frac{\tan\varphi'}{\sqrt{9+12\tan^2\varphi'}}, \quad K' = \frac{3c'}{\sqrt{9+12\tan^2\varphi'}} \qquad (7)$$

In finite-volume numerical analysis of the problem the Mohr-Coulomb (M-C) yield criterion is simulated using Drucker-Prager (D-P) yield criterion. The spatial domen is divided by a nonuniform grid consisting of 170 control volumes. The numerical analysis of the problem is also given in (Brebbia, 1984) using finite-element and boundary-element methods.

Numerical calculations of the maximum ground surface displacement are shown in Fig. 3. In this figure, finite-element and boundary element solutions applying M-C criterion, as well as finite-volume and boundary-element calculations using D-P criterion are given. It can be seen an excellent agreement of numerical solutions.

4. CONCLUSION

In this paper a finite volume based numerical formulation for predicting elasto-plastic behavior of solids obeying Drucker-Prager yield criterion is successfully applied. A segregated approach employed to solve resulting set of coupled algebraic equations shows very good convergence behaviour. The presented 2D steady case demonstrates very good accuracy of the method.

The future work will be focused on thoroughly investigation of the method application on a large class of limit load problems in geomachanics.

5. REFERENCES

Brebbia C.A.; Telles J.C.F. & Wrobel L.C. (1984). *Boundary Element Techniques*, Springer-Verlag, ISBN 0-387-12484-5, New York

Bijelonja, I (2002). *Finite* Volume method for analysis of small and large thermo-elasto-plastic deformation, Ph.D. thesis, University of Sarajevo, BiH

Demirdžić, I. & Martinović, D. (1993). Finite volume method for thermo-elasto-plastic stress analysis, *Computer Methods in Applied Mechanics and Engineering*, Vol.109, pp.331-349

Demirdžić, I. & Muzaferija , S. (1994). Finite volume method for stress analysis in comlpex domains, *Int. J. Numer. Methods Engrg.*, Vol.37, (1994) pp. 3751 – 3766, ISSN 1097-0207

Demirdžić, I. & Muzaferija, S. (1995). Numerical method for coupled fluid flow, heat transfer and stress analysis using unstructured moving meshes with cells of arbitrary topology, *Computer Methods in Applied Mechanics and Engineering*, Vol.125, pp.235-255