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ON PERIODIC SOLUTIONS AND ENERGY - CASIMIR MAPPING FOR MAXWELL-BLOCH EQUATIONS

LAZUREANU, C[ristian]; BINZAR, T[udor] & PATER, F[lavius]

Abstract: *In this paper we consider the real-valued Maxwell-Bloch equations on* $\mathbb{R}^3$*. Some connections between standard dynamical elements of considered system and some pure geometrical objects are presented. Thus, the existence of periodic solutions of the system, the image of the energy-Casimir mapping associated with the system and its connection with the stability of the equilibrium states are studied.*
Key words: *Maxwell-Bloch equations, stability, periodic orbits*

1. INTRODUCTION

The description of the interaction between laser light and a material sample composed of two-level atoms begin with Maxwell's equations of the electric field and Schrödinger's equations for the probability amplitudes of the atomic levels. The resulting dynamics is given by Maxwell - Schrödinger equations which have Hamiltonian formulation and moreover there exists a homoclinic chaos (Holm&Kovaćić, 1992).The modelling of chaotic dynamics restricted to $\mathbb{R}$ is described by the three dimensional real-valued Maxwell-Bloch equations:

$$\begin{cases} \dot{x} &= \quad y \\ \dot{y} &= \quad xz \\ \dot{z} &= -xy \end{cases} \tag{1}$$

The Hamilton-Poisson formulation, stability and control of system (1) have been studied in (David&Holm, 1992) and (Puta, 1994).

The aim of our paper is to present some dynamical and geometrical properties of system (1). A similar study for Rikitake System was presented in (Tudoran et al., 2009).

In section 2, by using the result of (Birtea et al., 2007) the existence of periodic solutions of system (1) is proved.

In section 3, we give the image of the energy-Casimir mapping and its connection with the stability of the equilibrium states of system (1) is described.

2. THE EXISTENCE OF THE PERIODIC SOLUTIONS

We recall that the equilibrium states of system (1) are given as the union of the following two families:

$$_1 = \{(M,0,0) \mid M \in \mathbb{R}\} \tag{2}$$

$$_2 = \{(0,0,M) \mid M \in \mathbb{R}\} \tag{3}$$

All the equilibrium states from the family $_1$ are nonlinearly stable and all equilibrium states from the family $_2$ are nonlinearly stable except for the equilibrium states $(0,0,M), M > 0$, which are unstable.

It is known that our dynamics (1) has the following Hamilton-Poisson realization $(\mathbb{R}^3, \Pi, H)$, where

$$\Pi = \begin{pmatrix} 0 & -z & y \\ z & 0 & 0 \\ -y & 0 & 0 \end{pmatrix} \tag{4}$$

is the Poisson structure and

$$H(x,y,z) = \frac{1}{2}x^2 + z \tag{5}$$

is the Hamiltonian function.

The center of the Poisson algebra $C^\infty(\mathbb{R}^3, \mathbb{R})$ is generated by the Casimir invariant

$$C(x,y,z) = \frac{1}{2}y^2 + \frac{1}{2}z^2. \tag{6}$$

Let us begin our study by searching for periodic solutions around nonlinearly stable equilibrium from the family $_2$.

The laws of dynamics (1) can be written in the form

$$\dot{u} = X(u) \tag{7}$$

where

$$u = (x,y,z)\,,\; X = y\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z} \tag{8}$$

The eigenvalues of the characteristic polynomial associated with the linearization at $(0,0,M)$ are $\lambda_1 = 0$, $\lambda_{23} = \pm i\sqrt{-M}$, $M < 0$ and the eigenspace corresponding to the eigenvalue zero has dimension 1:

$$V_{\lambda=0} = \operatorname{span}_{\mathbb{R}}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} \tag{9}$$

The restriction of the dynamics (1) to the coadjunct orbit

$$\mathcal{O}_{(0,0,M)} = \{(x,y,z) \in \mathbb{R}^3 \mid y^2 + z^2 = M^2\} \tag{10}$$

gives rise to a Hamiltonian system on a symplectic manifold. Considering the constant of motion

$$I(x,y,z) = \frac{1}{2}x^2 + z - \frac{1}{2M}(y^2 + z^2) \tag{11}$$

it follows that $\mathrm{d}I(0,0,M) = 0$ and $\mathrm{d}^2 I(0,0,M)|_{W\times W} > 0$, where $W = \ker \mathrm{d}C(0,0,M)$.

By applying the result of (Birtea et al., 2007) we have proved the following result:

Proposition. Let $(0,0,M) \in {}_2$ be such that $M < 0$. Then for each sufficiently small $\varepsilon \in \mathbb{R}$, any integral surface

$$\frac{x^2}{2\varepsilon^2} + \frac{y^2}{2(-M)\varepsilon^2} + \frac{(z-M)^2}{2(-M)\varepsilon^2} = 1 \tag{12}$$

contains at least one periodic solution of X whose period is close to $\frac{2\pi}{\sqrt{-M}}$, that is the period of the corresponding linear system around $(0,0,M)$.

Remark. The above periodic orbits are given as an intersection of two surfaces (see Fig.1):

$$\begin{cases} \frac{x^2}{2\varepsilon^2}+\frac{y^2}{2(-M)\varepsilon^2}+\frac{(z-M)^2}{2(-M)\varepsilon^2}=1 \\ \\ y^2+z^2=M^2 \end{cases} \tag{13}$$

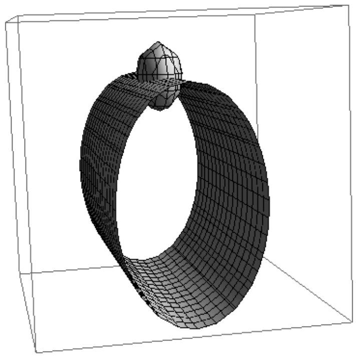

Fig. 1. Periodic orbits

3. THE IMAGE OF THE ENERGY-CASIMIR MAPPING

The goal of this section is to study the image of the Casimir mapping $Im(\mathcal{EC})$ associated with the Hamilton-Poisson realization of system (1). We are considering convexity properties of the image of the map $\mathcal{EC}$ as well as semialgebraic partitions of it and the associated Whitney canonical stratifications. For details on Whitney stratifications of semialgebraic manifolds, see, e.g. (Pflaum, 2001).

Recall that the energy-Casimir mapping $\mathcal{EC}\in C^\infty(\mathbb{R}^3,\mathbb{R}^2)$, is given by

$$\mathcal{EC}(x,y,z)=\left(\tfrac{1}{2}x^2+z\,,\ \tfrac{1}{2}y^2+\tfrac{1}{2}z^2\right) \tag{14}$$

The next proposition gives a characterization of the image of the energy-Casimir map $\mathcal{EC}$.

Proposition. The image of the energy-Casimir map is a convex subset of $\mathbb{R}^2$ given by the following:

$$Im(\mathcal{EC})=S_1\cup S_2 \tag{15}$$

where the sets $S_1,S_2\subset\mathbb{R}^2$ are semialgebraic manifolds given by

$$S_1=\left\{(h,c)\in\mathbb{R}^2|\ c\geq\tfrac{1}{2}h^2\right\} \tag{16}$$

$$S_2=\left\{(h,c)\in\mathbb{R}^2|\ c\leq\tfrac{1}{2}h^2,\ h\geq 0\right\} \tag{17}$$

Remark. The system (1) can be written in the following equivalent form:

$$\dot{u}=\nabla C(u)\times\nabla H(u) \tag{18}$$

Using this remark, one gets the following characterization for the equilibrium states of system (1).

Proposition. A point $u\in\mathbb{R}^3$ is an equilibrium state of system (1) if and only if u is a critical point of the energy-Casimir map $\mathcal{EC}$.

As any semialgebraic manifold has a canonical Whitney stratification, we identify this stratification and we connect it with the image through the energy-Casimir map of subsets of the families of equilibrium of system (1).

Proposition. The semialgebraic canonical Whitney stratifications of S_1, S_2 are given by the following:

$$S_1=Im\left(\mathcal{EC}|_{\substack{s\\2}}\right)\cup Im\left(\mathcal{EC}|_{\substack{u\\2}}\right)\cup\Sigma^p(S_1), \tag{19}$$

where $\Sigma^p(S_1)=\left\{(h,c)\in\mathbb{R}^2|\ c>\tfrac{1}{2}h^2\right\}$ is the principal stratum, and

$$S_2=Im\left(\mathcal{EC}|_{\substack{s\\1}}\right)\cup Im\left(\mathcal{EC}|_{\substack{u\\2}}\right)\cup\Sigma^p(S_2) \tag{20}$$

where $\Sigma^p(S_2)=\left\{(h,c)\in\mathbb{R}^2|\ c<\tfrac{1}{2}h^2, h>0\right\}$ is the principal stratum (see Fig.2).

We mention that the superscript s and u are used to point out the stable respectively unstable equilibrium state.

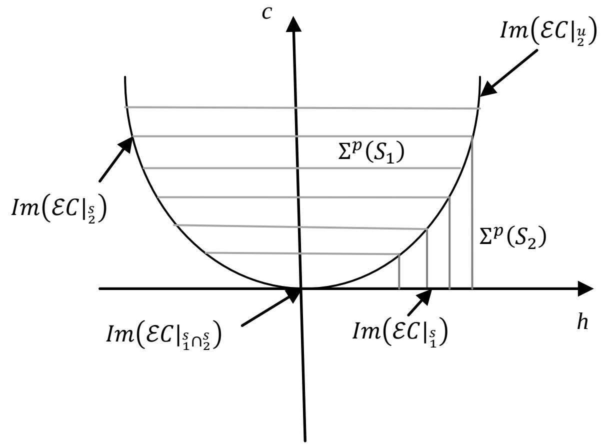

Fig. 2. Whitney canonical stratification of $Im(\mathcal{EC})$

Remark. As a convex set, the image of the energy-Casimir map is convexly generated by the images of the stable equilibrium of system (1), namely,

$$Im(\mathcal{EC})=\overline{co}\left\{Im\left(\mathcal{EC}|_{\substack{s\\1}}\right),Im\left(\mathcal{EC}|_{\substack{s\\2}}\right)\right\} \tag{21}$$

4. CONCLUSION

We have proposed to answer the following questions:

Are there any periodic solutions of our considered system around the stable equilibrium states? Is there any connection between the dynamical properties of Maxwell-Bloch equations and the geometry of the image of a vector valued constant of motion (the energy-Casimir mapping in our case) and if yes, how can one detect as many as possible dynamical elements and dynamical behaviour (stability in our case)?

The answers are given in sections 2 and 3.

In the future we will try to answer these questions for other dynamical systems.

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