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HOPF BIFURCATION ANALYSIS OF THE ECONOMICAL GROWTH MODEL WITH LOGISTIC POPULATION GROWTH AND DELAY

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Abstract: *In this paper, we formulate a new economical growth model with logistic population growth and delay. We study Hopf bifurcation of this growth model in which production occur with delay while new capital is installed (time-to-build). The time-to-build technology is shown to yield a system of differential functional equations with a steady state. We demonstrate that the steady state exhibits the Hopf bifurcation.*
Key words: *bifurcation, delayed, differential equations, economical growth*

1. INTRODUCTION

In this paper, we consider an economical growth model with logistic population growth in which production occurs with delay while new capital is installed. The optimality conditions, due to the introduction of the time delay, lead to a system of functional differential equations. We determine the steady state of this system and we investigate the local stability of the steady state by analyzing the corresponding transcendental characteristic equation of its linearized system. In the following, by choosing of the delay as a bifurcation parameter, we show that this model with a delay exhibits the Hopf bifurcation as in (Bundău, 2006). Therefore, the dynamics are oscillatory and this is entirely due to time-to-build production.

2. SETUP OF THE MODEL

Consider an economy that is inhabited by infinitely-lived households that, for simplicity, is normalized to one. Each household has access to a technology that transforms labor L and capital K into output Y by a neoclassical production function $F: R_+ \to R$.

We assume that at time t household use capital goods produced at time $t-\tau$, therefore the production at time t is given by

$$Y(t) = F(K(t-\tau), L(t)). \quad (1)$$

Denoting the capital per unit of labor by $k = K/L$ for any $L \neq 0$ we define the production function in intensive form as $f(k)$. Therefore, f is of class C^2, strictly increasing, strictly concave, linearly homogeneous, satisfying $f(0)=0$, and the Inada conditions $\lim_{k\to 0} f'(k) = \infty$, $\lim_{k\to\infty} f'(k) = 0$.

The representative household's preferences are represented by a continuous, strictly increasing and concave instantaneous utility function $U(c(t))$ and subject to discount rate ρ.

Considering the aggregate consumption $C(t)$, the capital accumulation equation is given by

$$\dot{K}(t) = F(K(t-\tau), L(t)) - \delta K(t-\tau) - C(t) \quad (2)$$

where $\delta \in [0,1]$ is the rate at which capital depreciates.

Following (Brida & Accinelli, 2007), the $L(t)$ is assumed to evolve according to the logistic law

$$\dot{L}(t) = aL(t) - bL^2(t), \quad (3)$$

with $a > b > 0$. For simplicity, the initial population has been normalized to one, $L_0 = 1$.

Using the properties of production function, we can rewritten the capital accumulation equation in intensive form, thus

$$\dot{k}(t) = f(k(t-\tau)) - (a - bL(t) + \delta)k(t-\tau) - c(t) \quad (4)$$

In this economy the representative household chooses at each moment in time the level of consumption $c(t)$ so that to maximize the global utility

$$\int_0^\infty U(c(t)) e^{-\rho t} dt \quad (5)$$

subject to the constraint (3), the budget constraint (4) and

$$k(t) = \varphi(t); t \in [-\tau, 0]; \quad (6)$$

where $0 < c(t) \leq f(k(t-\tau))$, $k(t-\tau)$ is the productive capital at time t, and $\varphi: (-\infty, 0] \to R_+$ is the initial capital function, it need to be specified in order to identify the relevant history of the state variable.

That economical problem, leads us to the following mathematical optimization problem (**P**).

Problem P. To determine (c^*, k^*, L^*) which maximizes the following functional $\int_0^\infty U(c(t)) e^{-\rho t} dt$ and which verifies

$$\begin{aligned} \dot{k}(t) &= f(k(t-\tau)) - (a - bL(t) + \delta)k(t-\tau) - c(t), \\ \dot{L}(t) &= aL(t) - bL^2(t), \\ k(t) &= \varphi(t); t \in [-\tau, 0], L(0) = L_0. \end{aligned} \quad (7)$$

To solve this optimization problem, we apply the generalized Maximal Principle for optimal control problems with delay. Analog to (Asea & Zac , 2000), the first order conditions of this model do not yield an advanced time argument because the co-state variable has the same timing, by convention, as the time the decision is made.

The first order conditions for the optimization problem (**P**) are

$$\begin{aligned} \dot{c}(t) &= \frac{U'(c(t))}{U''(c(t))}\left[\delta + \rho + a - bL(t) - f'(k(t-\tau))\right] \\ \dot{k}(t) &= f(k(t-\tau)) - (a - bL(t) + \delta)k(t-\tau) - c(t) \\ \dot{L}(t) &= aL(t) - bL^2(t) \end{aligned} \quad (8)$$

3. LOCAL STABILITY ANALYSIS AND HOPF BIFURCATION

Generally, the system (8) is not analytically solvable but we can show some qualitative properties of the solutions based on (Hale & Lunel, 1993), (Hassard et al, 1981). First, we determine the steady states (c^*, k^*, L^*) of the functional differential equations system (8), which are determined by setting $\dot{c}(t)=\dot{k}(t)=\dot{L}(t)=0$. From (8) it results

Proposition 3.1. (Stationary state) . The system of functional differential equations (8) has a unique steady state (c^*,k^*,L^*) which is determined by the following equations:

$$f'(k^*)=\delta+\rho\ ,\ c^*=f(k^*)-\delta k^*\ ,\ L^*=\frac{a}{b}\ . \tag{9}$$

With respect to the transformation

$$x_1(t)=c(t)-c^*\ ,\ x_2(t)=k(t)-k^*\ ,\ x_3(t)=L(t)-L^* \tag{10}$$

the system (8) becomes:

$$\begin{aligned}\dot{x}_1(t)&=F_1(x_1(t),x_2(t-\tau),x_3(t))\\ \dot{x}_2(t)&=F_2(x_1(t),x_2(t-\tau),x_3(t))\\ \dot{x}_3(t)&=F_3(x_3(t))\end{aligned} \tag{11}$$

where

$$F_1(x_1(t),x_2(t-\tau),x_3(t))=g(x_1(t)+c^*)[\rho+\delta+a-b(x_3(t)+L^*)-$$
$$-f'(x_2(t-\tau)+k^*)]$$

$$F_2(x_1(t),x_2(t-\tau),x_3(t))=f(x_2(t-\tau)+k^*)-$$
$$-\left(a-b(x_3(t)+L^*)+\delta\right)\left(x_2(t-\tau)+k^*\right)-(x_1(t)+c^*)$$

$$F_3(x_1(t),x_2(t-\tau),x_3(t))=a(x_3(t)+L^*)-b(x_3(t)+L^*)^2$$

$$g(x_1(t)+c^*)=\frac{U'(x_1(t)+c^*)}{U''(x_1(t)+c^*)}$$

Expanding F_1,F_2,F_3, given above, in Taylor series around of $0=(0,0,0)^T$ and neglecting the terms of order higher than three, we can rewrite the system (11) in the form

$$\dot{x}_1(t)=a_{010}x_2(t-\tau)+a_{001}x_3(x)+\frac{1}{2!}[a_{020}x^2{}_2(t-\tau)+2a_{110}x_1(t)x_2(t-\tau)+$$
$$+2a_{101}x_1(t)x_3(t)]+\frac{1}{3!}[a_{030}x^3{}_2(t-\tau)+3a_{201}x_1{}^2(t)x_3(t)+3a_{210}x_1{}^2(t)x_2(t-\tau)+$$
$$+3a_{120}x_1(t)x_2{}^2(t-\tau)]+...$$
$$\dot{x}_2(t)=b_{100}x_1(t)+b_{010}x_2(t-\tau)+b_{001}x_3(x)+\frac{1}{2!}[b_{020}x^2{}_2(t-\tau)+$$
$$+2b_{011}x_2(t-\tau)x_3(t)]+\frac{1}{3!}b_{030}x^3{}_2(t-\tau)... \tag{12}$$
$$\dot{x}_3(t)=c_{001}x_3(t)+\frac{1}{2!}c_{002}x^2{}_3(t)+...$$

where

$a_{010}=-g(c^*)f''(k^*)$, $a_{001}=-bg(c^*)$, $a_{020}=-g(c^*)f'''(k^*)$,

$a_{110}=-g'(c^*)f''(k^*)$, $a_{101}=-bg'(c^*)$, $a_{030}=-g(c^*)f^{(4)}(k^*)$,

$a_{201}=-bg''(c^*)$, $a_{210}=-g''(c^*)f''(k^*)$, $a_{120}=-g'(c^*)f'''(k^*)$,

$b_{100}=-1$, $b_{010}=f'(k^*)-\delta$, $b_{001}=bk^*$, $b_{020}=f''(k^*)$, $b_{011}=b$,

$b_{030}=f'''(k^*)$, $c_{001}=-a$, $c_{002}=-2b$.

To investigate the local stability of steady state we linearize the system (12). Let $u(t)=(u_1(t),u_2(t))^T$, be the linearized system variables, then the linearize system of (12) is given by

$$\dot{u}(t)=Au(t)+Bu(t-\tau) \tag{13}$$

where

$$A=\begin{bmatrix}0&0&a_{001}\\ b_{100}&0&b_{001}\\ 0&0&c_{001}\end{bmatrix},\ B=\begin{bmatrix}0&a_{010}&0\\ 0&b_{010}&0\\ 0&0&0\end{bmatrix}. \tag{14}$$

The associated characteristic equation of the liniarized system of the system (13) is given by:

$$\lambda^3-c_{001}\lambda^2-\left(b_{010}\lambda^2+(b_{100}a_{010}-c_{001}b_{010})\lambda-b_{100}a_{010}c_{001}\right)e^{-\tau\lambda}=0 \tag{15}$$

Proposition 3.2. *If $\tau=0$, then the characteristic equation (15) is given by*

$$-\lambda^3+(\rho-a)\lambda^2+(a\rho-a_{010})\lambda-a_{010}a=0 \tag{16}$$

The equation (16) has one positive eigenvalue and two eigenvalues with negative real part.

Proof: For to determine the sign of the real parts of the roots of equation (16), we make use of the following theorem.

Theorem. *The number of roots of the characteristic equation with positive real part is equal to the number of variations of sign in the scheme*

$$-1\ ,\ \ \rho-a\ ,\ \ a\rho-a_{010}-\frac{a_{010}a}{\rho-a}\ ,\ \ -a_{010}a\ . \tag{17}$$

This is an application of Routh-Hurwitz theorem.

In the case $\rho-a>0$, the sign in scheme (17) is $(-\ +\ +\ +)$.

In the case $\rho-a<0$, the sign of the quantities in scheme (17) can therefore be either $(-\ -\ +\ +)$ or $(-\ -\ -\ +)$.

In both circumstances exists only one change of sign, and therefore we have one positive eigenvalue and two eigenvalues with negative real part.

Next, we study the existence of Hopf bifurcation for system (8) by choosing the delay τ as the bifurcation parameter.

Proposition 3.3. *Let $\lambda=\lambda(\tau)$ be a solution of (15). If τ_c , ω are given by* $\omega=\frac{1}{\sqrt{2}}\sqrt{b_{010}^2+\sqrt{b_{010}^4+4a_{010}^2}}$, $\tau_c=\frac{1}{\omega}\arctan\frac{\rho\omega}{f''(k^*)g(c^*)}$ *and* $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)_{\lambda=i\omega,\tau=\tau_c}\neq 0$ *then a Hopf bifurcation occurs at the steady state,* (c^*,k^*,L^*) *when* τ *passes through* τ_c .

Proof: First, we would like to know when the equation (15) has purely imaginary roots $\lambda=\pm i\omega$ at $\tau=\tau_c$.

For $\lambda=i\omega$, from the equation (15) we obtain

$$-\omega^3-[g(c^*)f''(k^*)+a\rho]\omega\cos\omega\tau_c-[\rho\omega^2-ag(c^*)f''(k^*)]\sin\omega\tau_c=0$$
$$-a\omega^2+[\rho\omega^2-ag(c^*)f''(k^*)]\cos\omega\tau_c-[g(c^*)f''(k^*)+a\rho]\omega\sin\omega\tau_c=0$$

which implies that $tg(\omega\tau_c)=\frac{\rho\omega}{f''(k^*)g(c^*)}$, so that $\tau_c=\frac{1}{\omega}\arctan\frac{\rho\omega}{f''(k^*)g(c^*)}$. Differentiating the equation (15) with respect to τ, we obtain $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)_{\lambda=i\omega,\tau=\tau_c}\neq 0$.

4. CONCLUSIONS

In this paper, we formulate a growth model with delay for capital and with logistic population growth. Using the delay τ as a bifurcation parameter we have shown that a Hopf bifurcation occurs when this parameter passes through a critical value τ_c .

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