Annals of DAAAM for 2010 & Proceedings of the 21st International DAAAM Symposium, Volume 21, No. 1, ISSN 1726-9679
ISBN 978-3-901509-73-5, Editor B. Katalinic, Published by DAAAM International, Vienna, Austria, EU, 2010
Make Harmony Between Technology and Nature, and Your Mind will Fly Free as a Bird
Annals of DAAAM International

ANALYSIS OF DITHERED MEASUREMENT BASED ON WIDROW'S QUANTIZATION THEORY

KAMENSKY, M[iroslav] & KOVAC, K[arol]

Abstract: *Nonsubtractive dithering with averaging is common method for quantization error suppression in measurement systems with analog-to-digital converter (ADC). In this paper general theory of quantization is used for evaluation of theoretical root mean squared error (RMSE) for most usual types of additive noise. Theoretical dependence of RMSE on dither dispersion is presented. The analysis of influence of several stochastic signals upon the result of correction is proved by simulations.*
Key words: *moments of quantization error, nonsubtractive dithering*

1. INTRODUCTION

Striving for higher precisions in measurement technology naturally leads to methods, which can shift the resolution limit of the quantizer usually represented by the analog-to-digital converter (ADC) under the lowest significant bit (LSB). Dither is a random noise added to a signal prior to it (re)quantization in order to control the statistical properties of the quantization error (Wannamaker et al., 2000). Two dither types are distinguished. Term subtractive dither (SD) is used for the case, when the dither is subsequently subtracted from the output signal after quantization. Nonstubtractive dither (ND) is the second dither type and it is not subtracted from processed quantized signal. SD provides advantages, for which it is very often used in digital audio or video applications. Unfortunately SD is difficult to employ in many practical systems. But implementation of ND is much easier. So it could be used with advantages especially in measuring systems, where the root mean squared error (RMSE) of results is usually of interest. Here ND could yield to suppression of quantization error by mean value filtering of quantizer output. But theoretical analysis of ND should consider not only dither influence on mean error but also variance of result of digital filtering. Only mean error $E[\xi|s]=E[\varepsilon|s]$ used to be analyzed in some literature for exhibition of dither contribution like in (Carbone & Petri, 1994). We will show how to obtain better theoretical description coming out of general quantization theory (Widrow & Kollar, 2008) to get model of error corresponding to model applied in (Skartlien & Øyehaug, 2005) for Gaussian dither. Then several dither distributions will be discussed.

2. NONSUBTRACTIVE DITHERING

Schematic of considered system with nonsubtractive dithering is shown in the Fig. 1. The quantizer input is denoted as $w=s+d$, where s is measured value and d is added noise. Without dither the output of ideal ADC has quantization error

$$e = Q(s) - s = Q_{\mathrm{e}}(s) \tag{1}$$

Q(s) is transfer function of ideal quantizer and Q_{e}(s) is error characteristics of known saw-tooth shape with zero mean and peak-to-peak value of one quantization step q. For general

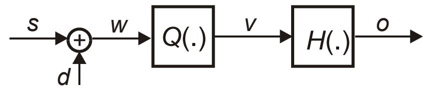

Fig. 1. Block structure of nonsubtractive dithering

system with ND (Fig.1) the quantizer output could be denoted as $v=Q(w)=Q(s+d)$ and the error after quantization is

$$\varepsilon = Q(s+d) - s = Q_{\mathrm{e}}(s+d) \tag{2}$$

Positive contribution of dither is lower peak-to-peak value of conditional mean $E[\varepsilon|s]$ compared to error e without dithering.

In ND application mean value should be evaluated to achieve suppression of quantization error. However it could be only estimated from finite number of samples N using filter H(.), Fig. 1. Very often simple averaging is applied for evaluation of output o, then error after averaging is $\xi = o-s$. Note that this error has additional stochastic part compared to $E[\varepsilon|s]$ if N is finite.

3. THEORY OF QUANTIZATION

Principles of the most general theory of quantization followed by some deeper analysis are described in (Widrow & Kollar, 2008). It uses approach called „area sampling" as the PDF's (probability density function) of quantizer input and output is related to each other through this special type of sampling. For given measured value s, the input PDF is smooth and the output PDF is discrete, because each input value is rounded towards the nearest allowable discrete level. The output PDF could be mathematically expressed by a string of Dirac delta functions. Also PDF of quantizer error ε could be described by means of a train of Dirac delta functions.

For the analysis of error properties the moments of error are of interest. E.g. in (Mariano & Ramos, 2006) the first and the second moment of quatization error were analyzed for sinusoidal input signal without dither. As we prefer evaluation of moments from characteristic function (CF) it is now favorable to write conditional CF of error ε resulting from area sampling process

$$\Phi_{\varepsilon|s} = \sum_{i=-\infty}^{\infty} \mathrm{sinc}\left(\frac{q}{2}(u+k\Psi)\right)\Phi_{\mathrm{d}}(u+k\Psi)e^{jk\Psi s} \tag{3}$$

where q is quantization step and $\Psi=2\pi/q$, Φ_{d} is CF of dither. From CF moments of error $E[\varepsilon^m|s]$ could be simply found using m-th derivatives of CF in u=0

$$E[\varepsilon^m|s] = \frac{1}{j^m}\frac{d\Phi_{\varepsilon|s}}{du}\bigg|_{u=0} \tag{4}$$

The first moment means conditional mean value $E[\varepsilon|s]$. In (Carbone & Petri, 1994) it was evaluated by other mathematical approach – at first generally and then also for several usual types of additive noise. We can prove these results from general theory of quantization using the first derivative of (3).

Assuming symmetrical dither with zero mean the CF Φ_d of dither should be real even function and then

$$E[\xi|s]=E[\varepsilon|s]=\sum_{k=1}^{\infty}\frac{q(-1)^{k}}{\pi k}\Phi_{d}(k\Psi)\sin(k\Psi s) \qquad (5)$$

To evaluate single parameter describing error of the whole range measurements it is better to use mean squared value

$$MSE_{T}(\infty)=\frac{1}{q}\int_{-q/2}^{q/2}E^{2}[\varepsilon|s]\,ds=\frac{q^{2}}{2\pi^{2}}\sum_{k=1}^{\infty}\frac{1}{k^{2}}\Phi_{d}^{2}(k\Psi) \qquad (6)$$

Equation (6) has meaning of the mean squared value of error ξ after averaging for the case of infinite number of averaged samples ($N=\infty$). Therefore this parameter still doesn't involve dispersion of error ξ for finite N.

4. ERROR OF ND WITH AVERAGING

For the evaluation of second moment of error ξ after averaging (Carbone, 1997) or (Skartlien & Øyehaug, 2005) use relation between variance of random value and variance of averaged random value. Second moment then could be solved from variance and mean. Finally total RMSE evaluated as mean of $E[\xi^2|s]$ is (Carbone, 1997)

$$\begin{aligned}RMSE_{T}^{2}(N)=&\ \frac{1}{N}\frac{1}{q}\int_{-\frac{q}{2}}^{\frac{q}{2}}\mathrm{Var}(\varepsilon|s)\,ds+\frac{1}{q}\int_{-\frac{q}{2}}^{\frac{q}{2}}E^{2}[\varepsilon|s]\,ds=\\ &=\frac{RMSE_{T}^{2}(1)}{N}+\frac{N-1}{N}MSE_{T}(\infty)\end{aligned} \qquad (7)$$

In (Skartlien & Øyehaug, 2005) it was derived that without averaging ($N=1$) it holds $RMSE_T^2(1)=\sigma_d^2+q/12$, where σ_d is standard deviation (STD) of dither, which could be proved by deduction from second derivative of (3). Then the final error parameter is

$$RMSE_{T}^{2}(N)=\frac{12\sigma_{d}^{2}+q^{2}}{12N}+\frac{N-1}{N}\frac{q^{2}}{2\pi^{2}}\sum_{k=1}^{\infty}\frac{1}{k^{2}}\Phi_{d}^{2}(k\Psi) \qquad (8)$$

We can now apply theory for the most usually used dither distributions. In the Tab. 1 CFs of dithers are listed, which could be simply substituted in (8) to get total RMSE. Then for some types of noise and for sufficient σ_d it is enough to keep only the first term in the series expansion, e.g. as shown in (Skartlien & Øyehaug, 2005) for Gaussian dither. In the Fig. 2 dependency of RMSE on σ_d for $N=10$ is depicted for dither distribution: a) triangular; b) sinusoidal; c) binary; d) 3-level discrete ($0,+D,-D$). Black theoretical lines are achieved by (8) considering only the first term of the series expansion. Dots represent simulation results obtained from 20 averaging processes ($N=10$) in 20 points of measured input s equally spread within one quantization step. As could be seen from the Fig. 2, in the case of triangular dither theoretical curve is very well fitting simulation results except short area of small dither STD. But especially for binary dither visible shift of theoretical curve near to minimal RMSE could be noticed caused by simplification of (8). Considering 10 terms of series expansion the result would be well corresponding for all analyzed noises.

Dither	$\Phi(k\Psi)$	D
Gaussian	$\exp(-2(k\sigma_d\pi/q)^2)$	∞
Uniform	$\mathrm{sinc}(kD\pi/q)$	$\sqrt{12}\sigma_d$
Triangular	$\mathrm{sinc}^2(kD\pi/(2q))$	$\sqrt{24}\sigma_d$
Sinusoidal	Bessel-0: $J_0(kD\pi/q)$	$\sqrt{8}\sigma_d$
Binary	$\cos(kD\pi/q)$	$2\sigma_d$
Discrete-3	$[1+2\cos(kD\pi/q)]/3$	$\sqrt{6}\sigma_d$

Tab. 1.CF of usual dither types (D means peak-to-peak value of noise)

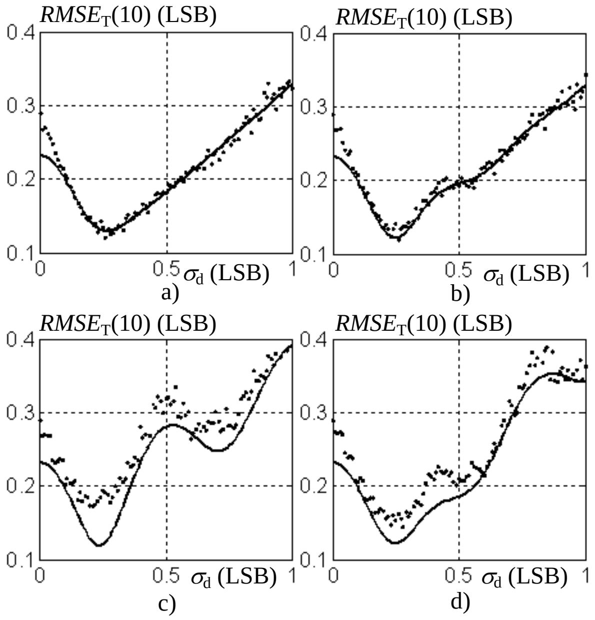

Fig. 2. Dependency of total RMSE on STD of dither

5. CONCLUSION

Theory of nonsubtractive dithering has been discussed in the paper from the general quantization theory point of view using characteristic functions. For measurement application theoretical root mean squared error (RMSE) has been considered as evaluating parameter of nonsubtractive dithering contribution. Sequence of mathematical steps needed for evaluation of total RMSE of averaged samples has been presented. Finally RMSE has been expressed for mostly used dither distributions. From comparison of theoretical RMSE with simulation it could be seen, that simplified error model is not sufficiently precise for some distributions of dither.

6. ACKNOWLEDGEMENTS

Work presented in this paper was supported by grant No. 2003SP200280802 and by VEGA grant No. 1/0551/09.

7. REFERENCES

Carbone, P. (1997). Quantitative Criteria for Design of Dither-Based Quantizing Systems. *IEEE Trans. on Instr. and Meas.*, Vol. 46, No. 3, June 1997, pp. 656-659, ISSN 0018-9456

Carbone, P. & Petri, D. (1994). Effect of Additive Dither on the Resolution of Ideal Quantizers. *IEEE Trans. on Instr. and Meas.*, Vol. 43, No. 3, June 1994, pp. 389-396, ISSN 0018-9456

Mariano, J. L. & Ramos, H. G. (2006). Validity of Widrow's model for sinusoidal signals. Measurement, Vol. 39, 2006, pp. 198-203, ISSN 0263-2241

Skartlien, R. & Øyehaug, L. (2005). Quantization error and Resolution in Ensemble Averaged Data With Noise. *IEEE Trans. on Instr. and Meas.*, Vol.54, No.3, June 2005, pp. 1303-1312, ISSN 0018-9456

Wannamaker, R. A.; Lipshitz, S. P.; Vanderkooy, J. & Wright, J. N. (2000). A Theory of Nonsubtractive Dither. *IEEE Trans. on Signal Processing*, Vol.48, No.2, February 2000, pp. 499-516, ISSN 1053-587X

Widrow, B. & Kollár, I. (2008). *Quantization Noise: Roundoff Error in Digital Computation, Signal Processing, Control, and Communications*, Cambridge University Press, ISBN 0521886716, Cambridge