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A THEOREM ON DISCRETE-TIME SCALE IN THE QUALITATIVE THEORY OF STOCHASTIC EVOLUTION EQUATION

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Abstract: *The case of stochastic evolution equation is a hot topic in (applied) mathematics and it has been studied by many authors among which we can mention Curtain and Pritchard, Dawson and Miyahara. Taking into account recent interest in this area, we obtain in this paper a theorem on discrete-time scale that addresses concerns with respect to the long-time behavior of the stochastic cocycles. Our result generalizes well-known theorems obtained by Perron and Li.*
Keywords: *uniform exponential stability, stochastic semiflows*

1. INTRODUCTION

The problem of existence of semiflows for stochastic evolution equations is a non-trivial one, mainly due to the well-known fact that finite-dimensional methods for setting (even continuous) stochastic flows break down in the infinite-dimensional context of stochastic evolution equations. In particular, Kolmogorov's continuity theorem fails for random fields parametrized by infinite-dimensional Hilbert spaces.

For the case of linear stochastic evolution equations with finite-dimensional noise, a stochastic semiflow (i.e. a random evolution operator) was obtained by Bensoussan & Flandoli (1995). Recently, Mohammed, Zhang & Zhao (2006) detect the existence of stochastic cocycles generated by mild solutions of a large class of semilinear stochastic evolution equations. Continuing their work, we obtain a theorem on discrete-time scale that addresses concerns with respect to the asymptotic behavior of the stochastic cocycles.

Our approach follows the well-established line of results initiated by Perron (1930) and Li (1934) for the deterministic case, which concerns the problem of stability of a deterministic system $x'(t) = A(t)x(t)$ and its connection with the existence of bounded solutions of the inhomogeneous system $x'(t) = A(t)x(t) + f(t)$.

2. STOCHASTIC COCYCLES

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and let $\mathfrak{B}(\mathbb{X})$ be the Banach algebra of all linear and bounded operators acting from $\mathbb{X}$ into $\mathbb{X}$. By $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ we denote a complete filtered probability space (i.e. (Ω, F, P) is a complete probability space, $\{F_t\}_{t\geq 0}$ is an increasing families of σ-algebras, F_0 contains all P-null sets of F and $F_t = \bigcap_{s\geq t} F_s$ for every $t \geq 0$).

Definition 2.1. A **stochastic semiflow** on Ω is a random field $\varphi: \mathbb{R}_+ \times \Omega \to \Omega$ such that

(s_1) $\varphi(0, \omega) = \omega$, for all $\omega \in \Omega$;

(s_2) $\varphi(t + s, \omega) = \varphi(t, \varphi(s, \omega))$, for all $t, s \geq 0$, $\omega \in \Omega$.

Example 2.2. Let $\mathbb{X}$ be a separable real Hilbert space and consider Ω as the space (with the compact open topology) of all continuous paths $\omega: \mathbb{R}_+ \to \mathbb{X}$ with $\omega(0) = 0$. Let F_t be the σ-algebra generated by the set $\{\omega \to \omega(u): u \leq t\}$, for every $t \geq 0$, and let F be the Borel σ-algebra on Ω. If P is the Wiener measure on Ω, then the quadruplet $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ is the canonical complete filtered probability space with the Wiener motion $W(t, \omega) = \omega(t)$ for all $t \geq 0$ and $\omega \in \Omega$.

The map

$$\varphi: \mathbb{R}_+ \times \Omega \to \Omega \ , \ \varphi(t, \omega)(s) = \omega(t + s) - \omega(t) \qquad (1)$$

defines a stochastic semiflow on Ω.

Definition 2.3. Let $\varphi: \mathbb{R}_+ \times \Omega \to \Omega$ be a stochastic semiflow on Ω. The mapping $\Phi: \mathbb{R}_+ \times \Omega \to \mathfrak{B}(\mathbb{X})$ is said to be a **stochastic cocycle** (over the semiflow φ) if it satisfies

(c_1) $\Phi(0, \omega) = I$ (the identity on $\mathbb{X}$), for all $\omega \in \Omega$;

(c_2) $\Phi(t + s, \omega) = \Phi\big(t, \varphi(s, \omega)\big)\Phi(s, \omega)$, for all $t, s \geq 0$, and $\omega \in \Omega$.

If in addition, there exist $M, \lambda > 0$ such that

(c_3) $E\|\Phi(t, \cdot)x\|^2 \leq Me^{\lambda(t-s)}E\|\Phi(s, \cdot)x\|^2$, for all $t, s \geq 0$, and $x \in \mathbb{X}$,

then Φ is a **stochastic cocycle with exponential growth in mean square.**

Example 2.4. Consider again the complete filtered probability space introduced in Example 2.2 and let $\{\mathcal{W}(t)\}_{t\geq 0}$ be an $\mathbb{X}$-valued Brownian motion with a separable covariance Hilbert space $\mathbb{H}$.

As usual, $\mathfrak{B}(\mathbb{H}, \mathbb{X})$ is the Banach space of all bounded linear operators from $\mathbb{H}$ into $\mathbb{X}$, while $\mathfrak{S}(\mathbb{H}, \mathbb{X}) \subset \mathfrak{B}(\mathbb{H}, \mathbb{X})$ is the subspace of all Hilbert-Schmidt operators $S: \mathbb{H} \to \mathbb{X}$ endowed with the norm

$$|S| = \left(\sum_{k=1}^{\infty} \|S(e_k)\|^2\right)^{1/2} , \qquad (2)$$

where $\{e_k\}_{k\geq 1}$ is a complete orthonormal system on $\mathbb{H}$.

Next, consider the **linear stochastic evolution equation**

$$du(t, x, \cdot) = Au(t, x, \cdot)dt + Bu(t, x, \cdot)d\mathcal{W}(t) \qquad (3)$$

where $A: D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, and $B: \mathbb{X} \to \mathfrak{S}(\mathbb{H}, \mathbb{X})$ is a bounded linear operator.

Assume that B can be extended to a bounded linear operator $B: \mathbb{X} \to \mathfrak{B}(\mathbb{X})$ (denoted by the same symbol B), and that the series $\sum_{k=1}^{\infty} \|B_k^2\|_{\mathfrak{B}(\mathbb{X})}$ converge, where B_k is the bounded linear operator on $\mathbb{X}$ defined by $B_k(x) = (Bx)(e_k)$, $x \in \mathbb{X}$, $k \geq 1$.

A **mild solution** of the above stochastic evolution equation is given by the family of $\{F_t\}_{t\geq 0}$-adapted processes $u(\cdot, x, \cdot): \mathbb{R}_+ \times \Omega \to \mathbb{X}$, $x \in \mathbb{X}$, satisfying the stochastic integral equation:

$$u(t, x, \cdot) = T(t)x + \int_0^t T(t - s)Bu(s, x, \cdot)d\mathcal{W}(s) \qquad (4)$$

Then, the mapping $\Phi: \mathbb{R}_+ \times \Omega \to \mathfrak{B}(\mathbb{X})$ defined by

$$\Phi(t, \omega)x = u(t, x, \omega) \qquad (5)$$

is a stochastic cocycle over the semiflow φ (see Example 2.2.)

3. EXPONENTIAL STABILITY IN MEAN SQUARE OF STOCHASTIC COCYCLES

In this section, we investigate a type of asymptotic behavior of a stochastic cocycle Φ, namely the exponential stability in mean square.

Definition 3.1. The stochastic cocycle $\Phi: \mathbb{R}_+ \times \Omega \to \mathfrak{B}(\mathbb{X})$ is said to be **exponentially stable in mean square** if there exist two positive constants N, ν such that

$$E\|\Phi(t,\ \cdot)x\|^2 \leq Ne^{-\nu(t-s)}E\|\Phi(s,\ \cdot)x\|^2 \qquad (6)$$

for all $t \geq s \geq 0$ and $x \in \mathbb{X}$.

Define $\mathcal{C}$ as the space of all $\mathbb{X}$-valued stochastic processes α with

$$\sup_{n\in\mathbb{N}}(E\|\alpha(n)(\cdot)\|^2) < \infty \ . \qquad (7)$$

Next, for $\alpha \in \mathcal{C}$ set

$$(\Gamma\alpha)(n)(\cdot) = \sum_{j=0}^{n} \Phi\big(n-k, \varphi(k,\cdot)\big)\alpha(\cdot) \ , \qquad (8)$$

for every $n \in \mathbb{N}$.

Obs. In what follows, we will often use $E\|\alpha(n)\|^2$, $E\|(\Gamma\alpha)(n)\|^2$ instead of $E\|\alpha(n)(\cdot)\|^2$, $E\|(\Gamma\alpha)(n)(\cdot)\|^2$, respectively.

Condition A. There exists some positive constant K such that

$$\sup_{n\in\mathbb{N}}(E\|(\Gamma\alpha)(n)\|^2) \leq \mathrm{K} \sup_{n\in\mathbb{N}}(E\|\alpha(n)\|^2) \ , \qquad (9)$$

for every stochastic process $\alpha \in \mathcal{C}$.

Theorem 3.2. *Let Φ be a stochastic cocycle with uniform exponential growth in mean square satisfying Condition A. Then, Φ is exponentially stable in mean square.*

Proof. Let $x \in \mathbb{X}$, $s \geq 0$ and set $n_0 = [s]+1$, where $[s]$ denotes the largest integer less than or equal with s. Consider the stochastic process

$$\alpha(n) = \begin{cases} \dfrac{\Phi(n_0,\cdot)x}{(E\|\Phi(n_0,\ \cdot)x\|^2)^{\frac{1}{2}}} \ , & n = n_0 \\ 0 \ , & n \neq n_0 \end{cases} . \qquad (10)$$

We have that $\sup_{n\in\mathbb{N}}(E\|\alpha(n)\|^2)$=1 and for each $n \geq n_0$,

$$(\Gamma\alpha)(n) = (E\|\Phi(n_0,\cdot)x\|^2)^{-\frac{1}{2}}\Phi(n,\cdot)x \ . \qquad (11)$$

By the hypothesis, we obtain that $E\|(\Gamma\alpha)(n)\|^2 \leq \mathrm{K}$, which implies that

$$E\|\Phi(n,\cdot)x\|^2 \leq \mathrm{K}\, E\|\Phi(n_0,\cdot)x\|^2 \ , \qquad (12)$$

for each $n \in \mathbb{N}$ and $x \in \mathbb{X}$.

Now, let $m \in \mathbb{N}$ and consider the stochastic process

$$\beta(n) = \begin{cases} \dfrac{\Phi(n,\cdot)x}{(E\|\Phi(n_0,\ \cdot)x\|^2)^{\frac{1}{2}}} \ , & n_0 \leq n \leq n_0+m \\ 0 \ , & n \neq n_0 \end{cases} . \qquad (13)$$

Clearly, $\sup_{n\in\mathbb{N}}(E\|\beta(n)\|^2) \leq$ K and

$$(\Gamma\beta)(j) = (j+1)(E\|\Phi(n_0,\cdot)x\|^2)^{-\frac{1}{2}}\Phi(j,\cdot)x \ , \qquad (14)$$

for every $j \geq n_0$. It follows that

$$\begin{aligned} \frac{(m+1)(m+2)}{2}\ \frac{E\|\Phi(n_0+m,\cdot)x\|^2}{E\|\Phi(n_0,\cdot)x\|^2} &\leq \mathrm{K}\sum_{j=n_0}^{n_0+m}(j+1)\frac{E\|\Phi(j,\cdot)x\|^2}{E\|\Phi(n_0,\cdot)x\|^2} \\ &\leq \mathrm{K}\sum_{j=n_0}^{n_0+m} E\|(\Gamma\beta)(j)\|^2 \\ &\leq \mathrm{K(m+1)}\ \sup_{j\in\mathbb{N}}(E\|(\Gamma\beta)(j)\|^2) \\ &\leq \mathrm{K^3(m+1)} \ , \qquad (15) \end{aligned}$$

which implies

$$E\|\Phi(n_0+m,\cdot)x\|^2 \leq \frac{2\ \mathrm{K}^3}{m+2}\ E\|\Phi(n_0,\cdot)x\|^2 \ . \qquad (16)$$

Therefore, there exist $k \in \mathbb{N}$ and $\eta \in (0,1)$ such that

$$E\|\Phi(n_0+k,\cdot)x\|^2 \leq \eta\ E\|\Phi(n_0,\cdot)x\|^2 \ , \qquad (17)$$

for all $n_0 \in \mathbb{N}$ and $x \in X$.

Let $t \geq 0$ and take $n = \left[\frac{t-s}{k}\right]$. Then, we can write down

$$\begin{aligned} E\|\Phi(t,\cdot)x\|^2 &\leq M^2e^{\lambda(k+1)}\ E\|\Phi(n_0+nk,\cdot)x\|^2 \\ &\leq M^2e^{\lambda(k+1)}\ \eta^n\ E\|\Phi(n_0,\cdot)x\|^2 \\ &\leq M^3e^{\lambda(k+2)}\ \eta^n\ E\|\Phi(s,\cdot)x\|^2 \qquad (18) \end{aligned}$$

Set now $\nu := -\frac{1}{k}\ln\eta$ and $N := M^3e^{\lambda(k+2)}\eta^{-1}$ to obtain

$$E\|\Phi(t,\ \cdot)x\|^2 \leq Ne^{-\nu(t-s)}E\|\Phi(s,\ \cdot)x\|^2 \ , \qquad (19)$$

for all $t \geq s \geq 0$ and $x \in \mathbb{X}$.

4. CONCLUSIONS

Theorem 3.2 can be seen as a sufficient condition for the exponential stability in mean square of the mild solutions of a stochastic evolution equation, such as (3). This condition is an extension of the so-called notion of admissibility, firstly used by Perron (1930) in the case of deterministic differential equations.

A possible extension of this paper concerns Condition A. We are looking to lessen this assumption by defining:

Condition A'. The stochastic process $\Gamma\alpha$ belongs to $\mathcal{C}$, for every stochastic process $\alpha \in \mathcal{C}$.

If we replace Condition A by Coondition A' in Theorem 3.2, does the conclusion remain valid?

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