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ON A STABILITY OF A QUASIPOLYNOMIAL

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Abstract: *The Laplace transform of a differential equation describing a system which contains delays in feedback loops results in a ratio of quasipolynomials, instead of obligatory polynomials as for delayless systems. Quasipolynomials can be then expressed as a linear combination of products of delay (exponential) terms and s-powers. The role of transfer function poles and that of the characteristic quasipolynomial is the same as in the traditional case. This paper utilizes the argument principle (the Mikhaylov criterion) in order to study stability properties of a selected quasipolynomial. Upper and lower bounds for a free real parameter are found via lemmas and theorems which are not proven due to the limited space. The obtained results are examined by a simulation example.*
Key words: *Quasipolynomial, stability analysis, delay systems*

1. INTRODUCTION

A generic feature of many real-life systems and processes is the existence of delays or latencies in their dynamics. The mathematical description of such systems can yield a model in the form of (an ordinary or rather a functional) linear differential equation (Bellmann & Cooke, 1963) in the neighborhood of an operating point. Direct application of the Laplace transform results in a transfer function as a ratio of so generalized polynomials, so called quasipolynomials (El'sgol'ts & Norkin, 1973) which are characterized as a linear combination of products of delay (exponential) terms and *s*-powers. Its spectrum is determined by roots of the denominator (as usual) and it has some interesting features, e.g. the number of poles is infinite. Similarly, even if conventional input-output process delay appears only, the characteristic closed loop equation is in the form of a quasipolynomial rather then a polynomial. This problem is usually solved by a rational approximation of exponential terms which is based e.g. on the McLaurin series expansion (Prokop & Corriou, 1997). Unfortunately, one can thus loose a part of system dynamics information.

For stability analysis, an affable feature of a class of quasipolynomials represented by the studied one is the fact that the argument principle (i.e. the Mikhaylov criterion) holds (e.g. in Górecki et al., 1989; Kolmanovskii & Myshkis, 1992). In this contribution, a selected (retarded) quasipolynomial is studied via unproven lemmas, observations and theorems. This quasipolynomial can represent a characteristic one of the closed loop system when control system with input-output and internal delays by a proportional controller. The aim is to find lower and upper bounds for a selectable real parameter such that the quasipolynomial is stable. The information about the limits can serve engineers to decide quickly about the stability or to set the free parameter properly.

The Mikhaylov criterion can be utilized for and limited to stability analysis of other retarded quasipolynomials as well. The obtained results are then verified using a simulation demonstrative example. Although the presented analysis is quite detailed, some statements have been deduced without proving them, which is the task to be solved in the future.

2. RETARDED QUASIPOLYNOMIAL

The selected retarded quasipolynomial which can represent the characteristic quasipolynomial of a feedback control system reads

$$m(s) = s + br\exp(-\tau s) + a\exp(-\vartheta s) \tag{1}$$

where $a,\ r \neq 0 \in \nabla,\ b, \tau, \vartheta > 0 \in \nabla$. The task is to find upper and lower bounds for *r* which stabilizes (1). This problem can be solved in geometric-like way using the principle argument which holds for quasipolynomial (1) as well. According to this principle, quasipolynomial (1) has all roots in the open left-half complex plane iff

$$\Delta \underset{s=\omega \mathrm{j}, \omega \in [0,\infty]}{\arg} m(s) = \frac{\pi}{2} \tag{2}$$

see e.g. (Myshkis, 1972). The whole following (here unproven) lemmas and theorems are based on the geometrical presentment of condition (2).

3. QUASIPOLYNOMIAL STABILITY FEATURES

Unproven statements about the stability properties of (1) follow.

Lemma 1: For $\omega = 0$, the imaginary part of the Mikhaylov curve of quasipolynomial (1) equals zero and it approaches infinity for $\omega \to \infty$.

Lemma 2: If (1) is stable, then the following inequality holds

$$r > \frac{-a}{b} \tag{3}$$

and thus the Mikhaylov curve starts on positive real axis.

Lemma 3: A point on the Mikhaylov curve lies in the first quadrant for an infinitesimally small $\omega = \Delta > 0$ iff

$$a\vartheta + br\tau \leq 1 \tag{4}$$

This point lies in the fourth quadrant iff

$$a\vartheta + br\tau > 1 \tag{5}$$

Lemma 4: If *a*, *b*, *r* are bounded, then $\mathrm{Re}\{m(\mathrm{j}\omega)\}$ is bounded for all $\omega > 0$.

Proposition 1: If (3) and (4) are satisfied together, then

$$a(\vartheta - \tau) \leq 1 \tag{6}$$

Proposition 2: If the following inequality holds

$$a(\vartheta-\tau)>1 \qquad (7)$$

then the appropriate Mikhaylov plot of a stable quasipolynomial (1) passes the fourth quadrant first.

Proposition 3: There always exists an intersection of the Mikhaylov curve with the imaginary axis.

Definition 1: Let (3) holds. A *crossover frequency* ω_0 is an element of the set

$$\Omega_0 := \{\omega : \omega > 0, \mathrm{Re}\{m(\mathrm{j}\omega)\} = 0, \mathrm{Im}\{m(\mathrm{j}\omega)\} = 0\} \qquad (8)$$

for some *crossover gain* r_0. A crossover frequency, hence, has to satisfy

$$\omega_0 \cos(\tau\omega_0) = a(\sin((\vartheta-\tau)\omega_0)) \qquad (9)$$

The crossover gain r_0 can be calculated as

$$r_0 = \frac{\omega_0 - a\sin(\vartheta\omega_0)}{b\sin(\tau\omega_0)} \qquad (10)$$

Definition 2: Let (3) holds. The *critical frequency* ω_C is defined as

$$\omega_C := \min\left\{\omega : \omega > 0, \mathrm{Re}\{m(\mathrm{j}\omega)\} = 0, \mathrm{Im}\{m(\mathrm{j}\omega)\} = 0, \underset{s=\omega\mathrm{j},\omega\in[0,\omega_0]}{\Delta \arg m(s)} = 0, \underset{s=\omega\mathrm{j},\omega\in[\omega_0,\infty]}{\Delta \arg m(s)} = \frac{\pi}{2}\right\} \qquad (11)$$

for the corresponding *critical gain* r_C given by (10), where ω_C is placed instead of ω_0, and $a \neq 0, b, \tau, \vartheta > 0$. Obviously, $\omega_C \in \Omega_0$ and the critical frequency is the least crossover frequency for which the argument change is zero for $\omega \in [0, \omega_C]$ and consequently it equals $\pi/2$ for $\omega \in [\omega_C, \infty]$.

Theorem 1: If $\sin(\tau\omega_C) > 0$, then quasipolynomial (1) is stable iff

$$\frac{-a}{b} < r < \frac{\omega_C - a\sin(\vartheta\omega_C)}{b\sin(\tau\omega_C)} \qquad (12)$$

Contrariwise, if $\sin(\tau\omega_C) < 0$, then quasipolynomial (1) is stable iff

$$r > \frac{\omega_C - a\sin(\vartheta\omega_C)}{b\sin(\tau\omega_C)} \geq \frac{-a}{b} \qquad (13)$$

where ω_C is the critical frequency.

Remark 1: It is not always easy to check, mainly without displaying the Mikhaylov plot, whether a crossover frequency calculated by (9) is critical and thus whether it can be used in Theorem 1. Sometimes only the sufficient stability condition is searched for; in this case, it is possible to use the following finding. Obviously, if the Mikhaylov plot for r_0 does not crosses the negative imaginary semi-axis, then the crossover frequency is critical, i.e. $\omega_0 = \omega_C$ (if there is no less one). This gives rise to the sufficient condition for ω_C and, consequently, for the quasipolynomial stability according to (12) and (13).

Remark 2: Definition 2 and Theorem 1 suggest situations when the quasipolynomial stabilization by the suitable choice of *r* is not possible. These are the two unpleasant possibilities which can come into being:

1) If ω_C does not exist. Thus, although Ω_0 is non-empty set, it may not contain $\omega_0 = \omega_C$.
2) If *r* could not satisfy (12), i.e.

$$\frac{\omega_C - a\sin(\vartheta\omega_C)}{b\sin(\tau\omega_C)} \leq \frac{-a}{b} \qquad (14)$$

4. EXAMPLE

Consider a quasipolynomial

$$m(s) = s + r\exp(-1.1s) - 5\exp(-s) \qquad (15)$$

with a free parameter *r*. One can observe that $\omega_C = 0.953$ which gives $r_C = 5.803$, according to (10). Hence, Theorem 1 yields the stabilizing limits $5 < r < 5.803$. Choose r = 5.4, the corresponding Mikhaylov plot is displayed in Fig. 1.

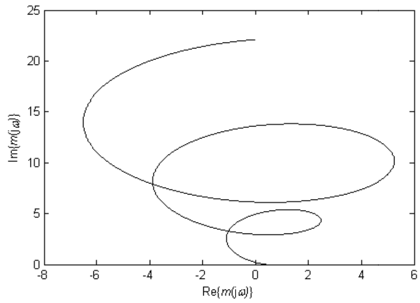

Fig. 1. The Mikhaylov plot of (15) for r = 5.4

5. CONCLUSION

The presented contribution has introduced some stability properties of a selected retarded quasipolynomial using the argument principle. The goal of the paper has been to specify the admissible limits for a selectable real parameter. Unproven statements have been presented because of the limited space. A simulation example figures the Mikhaylov plot of a stable quasipolynomial to demonstrate and verifies presented findings. The principle can be utilized for and limited to stability analysis of other retarded quasipolynomials as well.

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