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NEW SECULAR EQUATION OF RSPDT MATRIX AND ITS RATIONAL APPROXIMATION

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Abstract: *Cybenko and Van Loan (Cybenko & Van Loan, 1986) presented an algorithm which is a combination of bisection and Newton's method for the secular equation. This approach was improved considerably in (Mackens & Voss 1997) and (Kostic & Voss, 2002) by replacing Newton's method by a more appropriate root finding methods for the secular equation. In this note we present new secular function of real symmetric, positive definite Toeplitz matrix (RSPDT) T_n and theoretically construct its rational approximation.*
Key words: *eigenvalue problem, Toeplitz matrix, new secular equation*

1. INTRODUCTION

The problem of finding the smallest eigenvalue $\lambda_1^{(n)}$ of a real symmetric, positive definite Toeplitz matrix (RSPDT) T_n is of considerable interest in signal processing. Given the covariance sequence of the observed data, Pisarenko (Pisarenko, 1973) suggested a method which determines the sinusoidal frequencies from the eigenvector of the covariance matrix associated with its minimum eigenvalue. The computation of the minimum eigenvalue of T_n was studied in, e. g. (Cybenko & Van Loan, 1986; Kostic, 2004; Kostic & Cohodar, 2008 Mackens & Voss, 1997, 1998, 2000; Melman, 2006 Mastronardi, N & Boley, D.1999). Cybenko and Van Loan (Cybenko & Van Loan, 1986) presented an algorithm which is a combination of bisection and Newton's method for the secular equation. This approach was improved considerably in (Mackens & Voss 1997) and (Kostic & Voss, 2002) by replacing Newton's method by a more appropriate root finding methods for the secular equation. Taking advantage of the fact that the spectrum of a symmetric Toeplitz matrix can be divided into odd and even parts the methods based on the secular equation were accelerated in (Voss, 1999).

The paper is organized as follows. In Section 2 we present the basic properties of Toeplitz matrices and the notation we will use. In Section 3 we present the new method which is based on new secular equation and its approximation. In Section 4 we present conclusion and further research.

2. PRELIMINARIES

The $(i,j)^{th}$ element of an n x n symmetric Toeplitz matrix T_n is given by $t_{|i-j|}$ for some vector $(1,t_1,\ldots,t_{n-1})^T \in R^n$. The matrix T_n satisfies $JT_nJ= T_n$ and is therefore centrosymmetric. We use I for the identity matrix and J for the exchange, or "flip" matrix with ones on its southwest-northeast diagonal and zeros everywhere else. For simplicity's sake, our notation will not explicitly indicate the dimensions of the matrices I and J.

We note that for any $\lambda\in R$, the matrix $(T_n-\lambda I)$ is symmetric and centrossymmetric, whenever T_n is. In what follows, an important role is played by the so-called Yule-Walker equations. For an n x n symmetric Toeplitz matrix T_n, defined by $(1,t_1,\ldots,t_{n-1})$, this system of linear equations is given by $T_ny^{(n)}=-t^{(n)}$ where $t^{(n)}=(t_1,\ldots,t_n)^T$. There exist several methods to solve these equations. Durbin's algorithm solves them by recursively computing the solutions to lower-dimensional systems, provided all principal sub matrices are non-singular. This algorithm requires $2n^2+O(n)$ flops.

3. NEW METHOD

In this section we discuss the new method.
Let

$$T_n = \left(t_{|i-j|}\right)_{i,j=1,2,\ldots,n} \in R^{(n,n)} \tag{1}$$

be RSPDT matrix. We denote by $T_j \in R^{(j,j)}$ its j-th principal sub matrix, and we assume that its diagonal is normalized by $t_0 = 1$. If $\lambda_1^{(j)} \le \lambda_2^{(j)} \le \ldots \le \lambda_j^{(j)}$ are the eigenvalues of T_j then the interlacing property $\lambda_{j-1}^{(k)} \le \lambda_{j-1}^{(k-1)} \le \lambda_j^{(k)}, 2 \le j \le k \le n,$ holds.

In the determinant is defined the characteristic polynomial

$$\chi_n(\lambda) = det\begin{pmatrix} 1-\lambda & t_1 & t_2 & \cdots & t_{n-1} \\ t_1 & 1-\lambda & t_1 & \cdots & t_{n-2} \\ t_2 & t_1 & 1-\lambda & \cdots & t_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_1 & \cdots & 1-\lambda \end{pmatrix} =$$

$$= det\begin{pmatrix} A & B^T \\ B & T_{n-2}-\lambda I \end{pmatrix} \tag{2}$$

of matrix T_n where $A = \begin{pmatrix} 1-\lambda & t_1 \\ t_1 & 1-\lambda \end{pmatrix}$ and $B^T = \begin{pmatrix} t_2 & t_3 & \cdots & t_{n-1} \\ t_1 & t_2 & \cdots & t_{n-2} \end{pmatrix}$.

We denote $w^{(n-2)} = (t_2, t_3, \ldots, t_{n-1})^T$, $(T_{n-2}-\lambda I)z^{(n-2)} = -w^{(n-2)}$, $\tilde{y} = \left(y_2^{(n-2)}, y_3^{(n-2)}, \ldots, y_{n-2}^{(n-2)}\right)$ and $\tilde{t} = (t_2, t_3, \ldots, t_{n-2})$.

By matrix multiplication it is easy to get

$$\tilde{y} - y^{(n-3)} \cdot y_1^{(n-2)} = -(T_{n-3}-\lambda I)^{-1}\tilde{t}. \tag{3}$$

From last equation and

$$\begin{pmatrix} T_{n-3}-\lambda I & Jt^{(n-3)} \\ \left(t^{(n-3)}\right)^T J & 1-\lambda \end{pmatrix}\begin{pmatrix} \tilde{z} \\ z_{n-2} \end{pmatrix} = -\begin{pmatrix} \tilde{t} \\ t_{n-2} \end{pmatrix} \tag{4}$$

we get

$$z_{n-2} = -\frac{t_{n-1} + \left(t^{(n-3)}\right)^T J(\tilde{y} - y^{(n-3)}y_1^{(n-2)})}{1-\lambda + (t^{(n-3)})^T y^{(n-3)}}$$

$$\tilde{z} = Jy^{(n-3)}z_{n-2} + \tilde{y} - y^{(n-3)}y_1^{(n-2)} \tag{5}$$

For above proposed calculation we spent 6(n-3) flops. In equation (2) by using block elimination matrix B is eliminated. This way we get

$$\chi_n(\lambda) = det\begin{pmatrix} A - B^T(T_{n-2}-\lambda I)^{-1}B & B^T \\ 0 & T_{n-2}-\lambda I \end{pmatrix} \quad (6)$$

where

$$A - B^T(T_{n-2}-\lambda I)^{-1}B=$$
$$=\begin{pmatrix} 1-\lambda+\left(w^{(n-2)}\right)^T z^{(n-2)} & t_1+\left(w^{(n-2)}\right)^T y^{(n-2)} \\ t_1+\left(w^{(n-2)}\right)^T y^{(n-2)} & 1-\lambda+\left(t^{(n-2)}\right)^T y^{(n-2)} \end{pmatrix}. \quad (7)$$

We get following recursion

$$\chi_n(\lambda) = \chi_{n-2}(\lambda)\det(A - B^T(T_{n-2}-\lambda I)^{-1}B)$$

and we define new secular equation:

$$\det(A - B^T(T_{n-2}-\lambda I)^{-1}B) = 0$$

From modular decomposition of matrix $T_{n-2}-\lambda I$ we get

$$\lambda^2-2\lambda-(1-\lambda)\sum_{j=1}^{n-2}\frac{\alpha_j^2}{\lambda_j^{(n-2)}-\lambda}-$$
$$-2t_1\sum_{j=1}^{n-2}\frac{\beta_j}{\lambda_j^{(n-2)}-\lambda}+1-t_1^2 \quad (8)$$
$$+\sum_{i=1}^{n-2}\sum_{j=1}^{n-2}\frac{\gamma_i\delta_j}{(\lambda_i^{(n-2)}-\lambda)(\lambda_j^{(n-2)}-\lambda)}=0$$

and its rational approximation

$$g(\lambda) = \lambda^2-2\lambda+\frac{a\lambda+b-a}{c-\lambda}+1-t^2+\frac{e}{(c-\lambda)(d-\lambda)} \quad (9)$$

where a,b,c,d and e are determined such that:

$$\left(w^{(n-2)}\right)^T(T_{n-2}-\lambda I)^{-1}w^{(n-2)}+$$
$$+\left(t^{(n-2)}\right)^T(T_{n-2}-\lambda I)^{-1}t^{(n-2)}=\frac{a}{c-\lambda}$$

$$\left(w^{(n-2)}\right)^T(T_{n-2}-\lambda I)^{-2}w^{(n-2)}+$$
$$+(t^{(n-2)})^T(T_{n-2}-\lambda I)^{-2}t^{(n-2)}=\frac{a}{(c-\lambda)^2}$$

$$2t_1(w^{(n-2)})^T(T_{n-2}-\lambda I)^{-1}t^{(n-2)}=\frac{b-a}{c-\lambda}$$

$$2t_1(w^{(n-2)})^T(T_{n-2}-\lambda I)^{-2}t^{(n-2)}=\frac{b-a}{(c-\lambda)^2}$$

$$\left(w^{(n-2)}\right)^T(T_{n-2}-\lambda I)^{-1}w^{(n-2)}\left(t^{(n-2)}\right)^T(T_{n-2}-\lambda I)^{-1}t^{(n-2)}$$
$$-\left((w^{(n-2)})^T(T_{n-2}-\lambda I)^{-1}t^{(n-2)}\right)^2=\frac{e}{(c-\lambda)(d-\lambda)}$$

$$\left(w^{(n-2)}\right)^T(T_{n-2}-\lambda I)^{-2}w^{(n-2)}\left(t^{(n-2)}\right)^T(T_{n-2}-\lambda I)^{-1}t^{(n-2)}$$
$$+(w^{(n-2)})^T(T_{n-2}-\lambda I)^{-1}w^{(n-2)}(t^{(n-2)})^T(T_{n-2}-I)^{-2}t^{(n-2)}$$
$$-2(w^{(n-2)})^T(T_{n-2}-\lambda I)^{-1}t^{(n-2)}(w^{(n-2)})^T(T_{n-2}\lambda I)^{-2}t^{(n-2)}$$
$$=\frac{e(c+d-2\lambda)}{(c-\lambda)^2(d-\lambda)^2}$$

By making first derivation of odd equations respectively, we get even equations.

4. CONCLUSION

We have presented new secular equation of RSPDT matrix T_n and theoretically constructed its rational approximation. Our goal is to improve already existing algorithms which are based on secular equation (Mackens & Voss 1997) and (Kostic & Voss, 2002). With new algorithm we try to overcome the situation when minimal eigenvalues of matrix T_n and T_{n-1} are too close to each other. In this note, suggested approximation is convenient because coefficients a, b, c, d and e are easily computed and their computing does not require large number of flops. Through numerical experiments, which will be goal of future research, we will compare behaviour of new and old secular equation, as well as their rational approximations. By doing so, we will consider algorithm which has smaller number of flops and which is numerically more stable, to be the better one. In further research it is necessary to practically confirm suggested algorithm, compare it with previous algorithm and use symmetry properties of eigenvector.

5. REFERENCES

Cybenko, G. & Van Loan, C.F. (1986). Computing the minimum eigenvalue of a symmetric positive definite Toeplitz matrix, *SIAM J. Sci. Stat. Comput*, Vol. 7, No. 1, pp. 123-131. MRO819462(87b:65042)

Kostic, A. & Voss, H (2002). A method of order $1+\sqrt{3}$ for computing the smallest eigenvalue of a symmetric Toeplitz matrix. *WSEAS Transactions of Mathematics*, Vol. 1, pp. 1-7

Kostic A. (2004) *Verfahren zur Bestimmung einiger extremaler Eienwerte einer symmetrischen Toeplitz Matrix*, Shaker Verlag, ISBN 3-8322-3235-4, Aachen

Kostic & Cohodar (2008). Quadratic approximation of characteristic polynomial of symmetric positive definite Toeplitz matrix, Proceedings of 19th International DAAAM Symposium, ISBN 978-3-901509-68-1,ISSN 1726-9679, pp 192

Mackens , W. & Voss, H. (1998). A projection method for computing the minimum eigenvalue of a symmetric positive definite Toeplitz matrix. Lin.Alg.Appl. 275-276:401-415.

Mackens , W. & Voss, H. (2000). Computing the minimum eigenvalue of a symmetric positive definite Toeplitz matrix by Newton-type methods. *SIAM J. Sci. Comput.*, Vol. 21. No. 4, pp. 1650-1656 MR1756049 (2001g:65043)

Mackens , W. & Voss, H. (1997). The minimum eigenvalue of a symmetric positive definite Toeplitz matrix and rational Hermitian interpolSymtion. *SIAM J. Matr. Anal. Appl*, Vol. 18. pp. 523-534

Mastronardi, N & Boley, D.(1999). Computing the smallest eigenpair of a symmetric positive definite Toeplitz matrix. *SIAM J. Sci. Comput.* Vol. 20, No. 5, pp. 1921-1927. MR1694690

Melman, A. (2006). Computation of the Newton step for the even and odd characteristic polynomials of a symmetric positive definite Toeplitz matrix. *Mathematics of computation.* Vol. 75, No 254, pp. 817-832, S 025-5718(05)01796-5

Pisarenko, V. F. (1973) The retrieval of harmonics from a covariance function. *Geophys. J. R.. astr. Soc.* Vol. 33., pp. 347-366

Voss, H. (1999) Symmetric schemes for computing the minimum eigenvalue of a symmetric Toeplitz matrix. Lin.Alg.Appl.387:359-371