COMPARISON OF TRANSFER FUNCTION RATIONAL APPROXIMATIONS FOR LTI-TDS

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Abstract: Many controller design approaches for linear time-invariant time delay systems (LTI-TDS) require a delay approximation either in the state space or that of the transfer function. This paper is focused on the preliminary comparison of some selected easy-handling rational approximations of exponential elements in LTI-TDS transfer functions. Unlike traditional concept, internal (state) delays are considered as well. A program testing interface in Matlab-Simulink environment is developed.
Key words: time delay systems, Padé approximation, Laguerre shift, Kautz shift, Fourier analysis

1. INTRODUCTION

In recent decades a huge number of papers and works have been focused on model reduction of rational approximation of LTI-TDS, see e.g. (Makilla & Partington, 1999a; Makilla & Partington, 1999b; Battle & Miralles, 2000). A fair overview of some methods and approaches has been published in (Partington, 2004). However, a overwhelming majority of these methods deals with delays in input-output relation only ignoring state or internal delays on the left-hand side of differential equations, i.e. in the transfer function denominator (Zitek & Vitecek, 1999; Richard, 2003).

This paper, on the other side, focuses transfer function rational approximations for systems with internal delays since there are no theoretical results about approximations convergence and accuracy, and one can expect an interesting comparative results. When methods selection, emphasis is also put on handling with them; they must be easy to deal with so that anyone can use them effortlessly. The common principle of all approaches consists in substitution of exponential terms in the transfer function with rational fraction in Laplace complex variable s. Approaches are compared via H_∞ and H_2 norms.

For the authors intentions, the obtained results can be utilized e.g. when computer digital implementation of so called anisochronic controller obtained via algebraic controller design (Pekar & Prokop, 2008).

A Matlab-Simulink interface for testing all the applied methods is developed as well to make the comparison easier.

2. APPROXIMATION METHODS

In this chapter, used approaches and methods are described.

2.1 Padé Approximation

One of the most favorite and used rational approximations worldwide is the Padé one (Partington, 2004) given by the relation

$$\exp(-sT) \approx \frac{P(-s)}{P(s)}, P(s) = \sum_{k=0}^{n} \binom{n}{k} \frac{(2n-k)!}{(2n)!} (sT)^k \qquad (1)$$

2.2 Diagonal Padé Approximation

Diagonal Padé approximant reads (Battle & Miralles, 2000; Richard, 2003)

$$\exp(-sT) \approx \frac{P(s)}{P(-s)}, P(s) = \sum_{k=0}^{n} \frac{(2n-k)!}{k!(n-k)!} (-sT)^k \qquad (2)$$

One can, however, verify that (1) and (2) give the same resulting form.

2.3 Shift Operator Approximations

These methods are based on the fact that a delay term $\exp(-sT)$ expresses a shift operator and can be subjected to MacLaurin series expansion. A brief description of three the most important shift operators follows.

Laguerre shift. The eventual relation for this shift operator approximant reads

$$\exp(-sT) = \lim_{n \to \infty} \left(\frac{1 - \dfrac{sT}{2n}}{1 + \dfrac{sT}{2n}} \right)^n \qquad (3)$$

see e.g. (Makilla & Partington, 1999a).

Kautz shift. According to (Makilla & Partington, 1999a), the Kautz shift of the nth order is given by

$$\exp(-sT) = \lim_{n \to \infty} \left(\frac{1 - \dfrac{sT}{2n} + \dfrac{1}{2}\left(\dfrac{sT}{2n}\right)^2}{1 + \dfrac{sT}{2n} + \dfrac{1}{2}\left(\dfrac{sT}{2n}\right)^2} \right)^n \qquad (4)$$

For input-output delays only, Kautz shift is asymptotically (approximately) twice more accurate than the Laguerre one.

Padé shift. This type of shift approximation proceeds from the second order Padé approximation and can be expressed as

$$\exp(-sT) = \lim_{n \to \infty} \left(\frac{1 - \dfrac{sT}{2n} + \dfrac{1}{3}\left(\dfrac{sT}{2n}\right)^2}{1 + \dfrac{sT}{2n} + \dfrac{1}{3}\left(\dfrac{sT}{2n}\right)^2} \right)^n \qquad (5)$$

2.4 Fourier Analysis Based Method

This type of approximation was obtained from the analysis of the delay-in-feedback step response by the Fourier series expansion of the periodic part of the response. Using a negative feedback, one can obtain the following formula (Battle & Miralles, 2000)

$$\exp(-sT) = \frac{W(s)}{1 - W(s)}$$

$$W(s) = \frac{1}{2} - \frac{1}{4}Ts + \frac{2}{\pi^2}T^3 s^3 \sum_{k=0}^{n} \frac{1}{(2n+1)^2} \frac{1}{s^2 T^2 + (2n+1)\pi} \qquad (6)$$

3. APPROXIMATION ACCURACY MEASUREMENT

Transfer function norms instead of any complex analytic method for the comparison of approximation accuracy are used in this paper since this conception is sufficient for particular cases and for engineering practice.

3.1 H_2 Norm

The quadratic or H_2 norm of a stable strictly proper transfer function $G(s)$ reads

$$\|G\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 \, d\omega} \tag{7}$$

3.2 H_∞ Norm

This norm expresses so called L_2 gain and can be calculated as the supreme of the amplitude frequency characteristics

$$\|G\|_2 = \sup_{\omega} |G(j\omega)| \tag{8}$$

In (7) and (8) $G(s)$ is taken as the difference between the nominal and approximated transfer functions.

4. PROGRAM IMPLEMENTATION

All the methods described above have been implemented in a Matlab-Simulink program module with a user-friendly interface. The program enables a user to insert nominal $G(s)$, select approximation methods with appropriate orders and choose a norm or norms. It returns a graphical comparison of all chosen amplitude frequency responses, a table of calculated norms and, in the text form, we can obtain approximated transfer functions.

Due to the limited space, no graphics of the program interface are given here.

5. STUDY CASE

As mentioned above, the authors intent to use the comparison when selection of a suitable rational approximation of anisochronic (i.e. those with internal delays) controllers. This is one of possible steps of controllers' digitalization for the computer implementation.

Consider for example a first order unstable LTI-TDS

$$G_S(s) = \frac{0.6}{s - 0.2\exp(-0.8s)} \exp(-4s) \tag{9}$$

can be controlled by the controller of the transfer function

$$G_R(s) = \frac{3.78s + 0.9 - 0.33\exp(-0.8s)}{s + 2(1 - \exp(-4s))} \tag{10}$$

see details in (Prokop & Pekar, 2010).

Let $n \in \{1,2,3,4,5\}$ and test the accuracies for all methods introduced above. The best results for each of them are displayed in Tab 1.

Method	H_∞	n
Padé approx.	0.729	1
Laguerre shift	0.729	1
Kautz shift	0.694	4
Padé shift	0.82	1
Fourier analysis	1.134	5

Tab. 1. Comparison of rational approximations of (10)

Apparently, the Kautz shift of the 4th order provides the best result for the study case; however, Padé approximation and Laguerre shift even of the 1st order give comparable estimations which are much easier to utilize. Notice that for anisochronic systems, a higher order does not mean a better approximation in general.

H_2 norm can not be utilized since for non-strictly proper transfer functions reaches infinity.

6. CONCLUSION

In our contribution, we have focused some easy-handling transfer function rational approximations. Dealing with internal delays in controlled plants as well as controllers was the primary motivation for our work since no theoretical results were made for this purpose.

A GUI application has been developed in Matlab for comparison and calculating approximations of the delay systems. The application allows easy entering a plant transfer function, selecting methods to be compared and the choice of norms by which the accuracy of approximation is calculated. It is also possible to specify an order of the approximation for each selected method.

The main finding is that the accuracy of an approximation is not straightly dependent on its order in all cases, as holds in the case of input-output time delay.

7. ACKNOWLEDGEMENTS

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