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NUMERICAL ANALYSIS OF SECULAR FUNCTIONS OF A REAL SYMMETRIC POSITIVE DEFINITE TOEPLITZ MATRIX

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Abstract: *In this note we present numerical analysis of secular functions of a real symmetric positive definite Toeplitz matrix (RSPDTM). In paper (Kostic et al., 2011) is given general form of secular function of RSPDTM. From numerical analysis of secular functions emerged new algorithm which represents upgrading the cheapest algorithm by 14 percent or by 17 percent, respectively, when matrices have large dimension.*
Key words: *eigenvalue problem, Toeplitz matrix, secular function, numerical analysis*

1. INTRODUCTION

From Pisarenko´s work (Pisarenko, 1973) the problem of finding the smallest eigenvalue of a real symmetric, positive definite Toeplitz matrix (RSPDTM) plays an important role in signal processing. The computation of the minimum eigenvalue of T_n was studied in, e.g. in (Cybenko & Van Loan, 1986; Kostic, 2004; Kostic & Cohodar, 2008; Mackens & Voss, 1997, 2000; Melman, 2006; Mastronardi & Boley, 1999).

In this paper we bring criteria for comparison of different secular functions of RSPDTM and develop new method for calculating the smallest eigenvalue of the RSPDTM, which is based on using the smallest roots from two different secular functions.

The paper is organized as follows. In Section 2 we present the basic properties of Toeplitz matrices and the notation we will use. In Section 3 we present criteria for comparison, numerical analyses and new algorithm. In Section 4 we present conclusion of the paper.

2. PRELIMINARIES

The Teoplitz matrix is quadratic matrix which has the same elements in its respective diagonals, which means that $a_{ij} = a_{n+1-j,n+1-i}$. Because we take in consideration symmetric matrices it also means that $a_{ij} = a_{ji}$. The Teoplitz matrix is defined with vector $(1, t_1, \ldots, t_{n-1}) \epsilon R^n$ so the $(i,j)^{th}$ element of an n x n symmetric Toeplitz matrix T_n is given by $t_{|i-j|}$. We can conclude, form above given, that Teoplitz matrices are centrosymmetric and satisfy $JT_nJ= T_n$. We use I for the identity matrix and $J:= \left(\delta_{i,n+1-i}\right)_{i,j=1,\ldots,n}$ for the exchange, or "flip" matrix.

3. NUMERICAL ANALYSES AND CRITERIA FOR COMPARISON

Kostic, Velic & Bektesevic (Kostic et al., 2011) gave general form of the secular function of RSPDTM T_n:

$$q_k(\lambda) := \frac{p_n(\lambda)}{p_{n-k}(\lambda)} \tag{1}$$

where k<n, $p_n(\lambda)$ and $p_{n-k}(\lambda)$ are characteristic polynomials of RSPDTM T_n and T_{n-1} respectively.

In paper (Kostic et al., 2011) is given explained idea about pole shifting of secular function from the smallest root of the secular function. Secular functions given with (1) can be compared in several ways, e.g.: average distance between the root and the pole of secular function, modeling simplicity of secular function, or speed of the used algorithm on different secular functions.

To analyze secular functions of RSPDTM we considered the following class of Toeplitz matrices:

$$T_n = m \sum_{k=1}^{n} \eta_k \, T_{2\pi\theta_k}, \tag{2}$$

where m is chosen such that the diagonal of T_n is normalized to $t_0=1$. $T_\theta = \left(t_{i,j}\right) = (\cos(\theta(i-j)))$ and η_k and θ_k are uniformly distributed random in the interval [0,1] (Cybenko & Van Loan, 1986). We considered dimensions $n=32$, 64, 128, 256, 512 and 1024 of RSPDTM.

In case of using first criteria, when distance between the root and the pole of secular function of RSPDTM which we will denote with letter d for different k and dimension $n=32$ we get following table:

k	d
1	$5.032 \cdot 10^{-3}$
2	$8.9147 \cdot 10^{-3}$
3	$1.2703 \cdot 10^{-2}$
4	$1.85906 \cdot 10^{-2}$
5	$2.5839 \cdot 10^{-2}$
6	$3.39655 \cdot 10^{-2}$
7	$4.49125 \cdot 10^{-2}$
15	$1.77528 \cdot 10^{-1}$
16	$2.00584 \cdot 10^{-1}$
17	$2.2756 \cdot 10^{-1}$
28	$7.139 \cdot 10^{-1}$
29	$7.85097 \cdot 10^{-1}$

Tab. 1. The distance between the smallest root and the pole

However, even though the distance between the smallest root and the pole is very important, it is not absolutely crucial for the quality of the secular function.

Second criteria is also important because it depends of the modeling skills of given secular function. In paper (Kostic et al., 2010) we have seen one of the ways for modeling secular function for $k=2$.

We are sure we can use Newton's method on every secular function, because iteration

$$\lambda = \lambda - \frac{q_{n-k}}{q^{`}_{n-k}} \tag{3}$$

can easily be transformed to

$$\lambda = \lambda - \frac{1}{\frac{p`_n}{p_n} - \frac{p`_{n-k}}{p_{n-k}}} \quad (4)$$

whereas respective quotients can be calculated by using Melman's theorem (Melman, A. 2006).:

Theorem1. The Newton step for the characteristic polynomial $p_n(\lambda)$ of a n x n symmetric positive definite Toeplitz matrix T_n at $\lambda = \bar{\lambda} < \lambda_1$ is given by:

$$N(\bar{\lambda}) = \frac{q(\bar{\lambda})}{\sum_{j=1}^{n}(2j-n)w_j^2} \quad (5)$$

where q is as in Definition 1. and the first row of T_n is given by $(1,t^T)$ and $w = (w_1, \ldots, w_{n-1}) = -JN(T_{n-1} - \bar{\lambda}I)^{-1}t$. For compactness of writing, we have set w_n=1.

Table 2 contains the average number of Durbin steps, where we used Newton's method applied for different secular functions (different k), needed to determine each of the dimension *n=32, 64, 128, 256, 512* and *1024* in *100* test problems.

n=32		n=64		n=128		n=256	
k	steps	k	steps	k	steps	k	steps
1	8.17	1	9.83	1	10.63	1	12.15
2	5.97	2	6.82	2	8.09	2	9.31
3	5.77	3	5.94	3	6.89	3	8.41
4	5.96	4	5.84	4	6.52	4	7.69
5	6.32	5	6.28	5	6.26	5	7.03
6	6.64	6	6.59	6	6.32	6	6.81
15	7.88	31	8.869	63	9.07	127	10.2
16	7.92	32	8.89	64	9.09	128	10.2
17	7.96	33	8.92	65	9.09	129	10.2
28	8.33	60	9.41	120	9.33	252	10.39
29	8.36	61	9.41	121	9.33	253	10.39

Tab. 2. Average number of Durbin steps, where we used Newton's method applied for different secular functions (different k)

From the table can be seen that the best results are obtained for *n=32* and *k=3*; *n=64* and *k=4*; *n=128* and *k=5*; *n=256* and *k=6*. By analogy best results are obtained for *n=512* and *k=7*.

If we for given n and k combine given secular functions q_{n-k} and q_{n-1} in case when we are in front of the smallest root of the functions we get new algorithm which is significantly better than already existing algorithms. Here we use the fact that by using Newton's method on function q_{n-1} top boundary of the smallest eigenvalue is always obtained. By comparing the best algorithm for secular function without exploiting symmetry given in (Kostic, A. & Voss, H. 2002) we get

dim	new	best
32	5.39	4.34
64	5.19	5.14
128	5.22	5.25
256	5.52	5.84
512	5.65	6.62
1024	6.05	7.26

Tab. 3. Comparison of the new algorithm with the best existing algorithm (Kostic, A. & Voss, H. 2002)

It can be seen that improvement of 14 percent is obtained for dimension *n = 512* and improvement of 17 for dimension *n = 1024*. Now, it is clear that the biggest improvement of 17 percent is obtained for biggest matrix dimension.

4. CONCLUSION

In this paper we bring different criteria for comparison of different secular functions (different k). We made serious analysis using third criteria by applying Newton's method on different secular functions. If we combine two secular functions q_{n-k} and q_{n-1} and apply Newton's method on them we get new algorithm which enables us to get smallest eigenvalue of RSPDTM faster. We also used fact that by applying Newton's method on function q_{n-1} top boundary of the smallest eigenvalue is always obtained.

It would be interesting to consider root finding method for rational interpolation of secular functions q_{n-k} and q_{n-1}, which will be subject of our further research.

In further research we plan to use symmetry properties of RSPDTM, because symmetry properties should accelerate already existing algorithms.

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