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A (2,3)-PADE APPROXIMATION OF SECULAR FUNCTION OF TOEPLITZ MATRIX

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Abstract: *In this note we present a (2,3)-Padé approximation of secular function for computing the smallest eigenvalue λ_1 of a real symmetric positive definite Toeplitz matrix (RSPDTM). This method is based on approximation of secular function by using the Taylor series characteristic polynomial of R SPDTM The characteristic polynomial has been approximated with the polynomial of the third order from the Taylor series because it is easy to calculate $p''_{n-1}(\lambda)$ from specific structure of Toeplitz matrix from Gohberg-Simencul formulae. It is very easy to calculate $p'''_{n-1}(\lambda)$ from the secular function.*

Key words*: eigenvalue problem, Toeplitz matrix, (2,3)-Padé approximation*

1. INTRODUCTION

From Pisarenko´s work (Pisarenko, 1973) the problem of finding the smallest eigenvalue of a real symmetric, positive definite Toeplitz matrix (RSPDTM) plays an important role in signal processing. The computation of the minimum eigenvalue of T_n was studied in, e. g. in (Cybenko & Van Loan, 1986; Kostic, 2004; Kostic & Cohodar, 2008; Mackens & Voss, 1997, 2000; Melman, 2006; Mastronardi & Boley, 1999).

In this paper we propose a (2,3)-Padé approximation for the computation of the root of the secular function of a real symmetric positive definite Toeplitz matrix. This method is a combination of approximation for secular function and approximation for characteristic polynomial.

The paper is organized as follows .In Section 2 we present the basic properties of Toeplitz matrices and the notation we will use. In Section 3 we present (2,3)-Padé approximation for secular function. In Section 4 we present numerical results for Section 3. In Section 5 we present conclusion of the paper.

2. PRELIMINARIES

The Teoplitz matrix is quadratic matrix which has the same elements in its respective diagonals, which means that $a_{ij} = a_{n+1-j,n+1-i}$. Because we take in consideration symmetric matrices it also means that $a_{ij} = a_{ji}$. The Teoplitz matrix is defined with vector $(1, t_1, \ldots, t_{n-1}) \in R^n$ so the (*i,j*)th element of an *n* x *n* symmetric Toeplitz matrix T_n is given by $t_{|i-j|}$. We can conclude, form above given, that Teoplitz matrices are centrosymmetric and satisfy $JT_nJ = T_n$. We use I for the identity matrix and $J := (\delta_{i,n+1-i})_{i,j=1,\ldots,n}$ for the exchange, or "flip" matrix.

3. A (2, 3)-PADÉ APPROXIMATION

A Padé rational approximation to f(x) on [a,b] is the quotient of two polynomials $P_n(x)$ and $Q_m(x)$ of degrees n and m, respectively. We use the notation $R_{n,m}(x)$ to denote this quotient:

$$R_{n,m}(x) = \frac{P_n(x)}{Q_m(x)}$$

Padé approximation is based on theorem

Theorem (Padé Approximation)

Assume that $f \in C^{n+m}$, and that $f(x)$Maclaurin polynomial expansion of degree at least $n+m$. The

$$f(x) \approx R_{n,m}(x) = \frac{P_n(x)}{Q_m(x)}$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degree *n* and *m*, respectively.

Melman (Melman 1999) considered rational approximations

$$r_j(\lambda) = -1 + \lambda + \rho_j(\lambda)$$

of f where

$$\rho_1(\lambda) := \frac{a}{b-\lambda}, \quad \rho_2(\lambda) := a + \frac{b}{c-\lambda}, \quad \rho_3(\lambda) := \frac{a}{b-\lambda} + \frac{c}{d-\lambda}$$

And the parameters *a, b, c, d* are determined such that

$$\rho_j^{(k)}(0) = \frac{d^k}{d\lambda^k} t^T (T_{n-1} - \lambda I)^{-1} t\big|_{\lambda=0} = k! t^T T_{n-1}^{-(k+1)} t, k = 0,1,\ldots, j.$$

Thus ρ_1, ρ_2 and ρ_3, respectively, are the (0,1)-, (1,1)- and (1,2)- Padé approximations of $\Phi(\lambda) := t^T (T_{n-1} - \lambda I)^{-1} t$ (Melman, 1999).

Here we present a new rational approximation of the secular function. This new rational approximation is actually (2, 3)-Padé approximation for secular function.

The secular function $q(\lambda) = -1 + \lambda + t^T (T_{n-1} - \lambda I)^{-1} t$ may be written as

$$q(\lambda) = -\frac{p_n(\lambda)}{p_{n-1}(\lambda)} \tag{1}$$

where $p_n(\lambda)$ and $p_{n-1}(\lambda)$ are characteristic polynomials of T_n and T_{n-1} respectively.

We consider a rational approximation

$$r(\lambda) = d + \lambda - \bar{\lambda} + \frac{A_2(\lambda-\bar{\lambda})^2 + A_1(\lambda-\bar{\lambda}) + A_0}{B_3(\lambda-\bar{\lambda})^3 + B_2(\lambda-\bar{\lambda})^2 + B_1(\lambda-\bar{\lambda}) + B_0} \tag{2}$$

of *q*, where $d := \bar{\lambda} - 1$ and parameters A_2 and B_3 such that

$$r(\bar{\bar{\lambda}}) = q(\bar{\bar{\lambda}}) \qquad r'(\bar{\bar{\lambda}}) = q'(\bar{\bar{\lambda}}) \tag{3}$$

$\bar{\lambda}$ and $\bar{\bar{\lambda}}$ are not in the spectrum of T_n and T_{n-1}. Parameters B_0, B_1 and B_3 are given by to approximate characteristic polynomial of T_{n-1} with the polynomial of the second order based on the Taylor series. Parameters A_0 and A_1 can be easily calculated by dividing characteristic polynomials p_n and p_{n-1}. We give

$$B_0 = \frac{1}{q(\bar{\lambda})}$$

$$B_1 = -\frac{p'_{n-1}(\bar{\lambda})}{p_{n-1}(\bar{\lambda})} \cdot \frac{1}{q(\bar{\lambda})} \tag{4}$$

$$B_2 = \frac{p''_{n-1}(\bar{\lambda})}{2p_{n-1}(\bar{\lambda})} \cdot \frac{1}{q(\bar{\lambda})}$$

$$A_0 = 1 - \frac{d}{q(\bar{\lambda})}$$

$$A_1 = -\frac{p'_n(\bar{\lambda})}{p_n(\bar{\lambda})} - \frac{1}{q(\bar{\lambda})} + \frac{d}{q(\bar{\lambda})} \cdot \frac{p'_{n-1}(\bar{\lambda})}{p_{n-1}(\bar{\lambda})} \tag{5}$$

$$t = \bar{\bar{\lambda}} - \bar{\lambda}$$

$$B_3 = \frac{(B_1 t + 2B_0)\cdot\left(q(\bar{\bar{\lambda}}) - d - t\right) - (B_2 t^3 + B_1 t^2 + B_0 t)\left(q'(\bar{\bar{\lambda}}) - 1\right) - A_1 t -}{t^3 \cdot \left(q'(\bar{\bar{\lambda}})t + q(\bar{\bar{\lambda}}) - d - 2t\right)}$$

$$A_2 = \frac{\left(q(\bar{\bar{\lambda}}) - d - t\right)(B_3 t^3 + B_2 t^2 + B_1 t + B_0) - A_1 t - A_0}{t^2}$$

For easy calculation of given parameters we need theoretical base given in works (Kostic 2009, Kostic & Cohodar 2008 and Melman 2006).

4. NUMERICAL RESULTS

To test the method we considered the following class of Toeplitz matrices:

$$T_n = m\sum_{k=1}^{n} \eta_k T_{2\pi\theta_k}, \tag{6}$$

where m is chosen such that the diagonal of T_n is normalized to t_0=1. $T_\theta = (t_{i,j}) = (\cos(\theta(i-j)))$ and η_k and θ_k are uniformly distributed random in the interval [0,1] (cf. Cybenko and Van Loan). Table 1 contains the average number of Durbin steps needed to determine in 100 test problem each of the dimension n=32,64,128,256,512 and 1024.

Dim	New method
32	4.8575
64	5.29
128	5.2925
256	5.72
512	6.38
1024	6.8

Tab. 1. The average number of Durbin steps

5. CONCLUSION

In this paper we present a (2,3)-Padé approximation of the secular function for computing the smallest eigenvalue λ_1 of a real symmetric positive definite Toeplitz matrix (RSPDTM) . This method is based on approximation of secular function by using the Taylor series characteristic polynomial of RSPDTM. In this way, we improved previously developed algorithm what we also have expected, because we used second derivative of characteristic polynomial. Calculation of second derivative of characteristic polynomial is interesting result witch enables new rational approximation for secular function. New rational approximation presented in this note is actually (2,3) – Padé approximation. It is proven that the Padé approximation is the best approximation of a rational function. In this paper we have also numerically check this contention. This is the main contribution of the paper.

It is very interesting to approximate the even and odd secular functions on above pointed out manner but with much more using property of symmetry of the Toeplitz matrices. This will be further investigated in the next step of our future research plans.

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